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The compatible mapping of type (P) in intuitionistic N- fuzzy metric space with applications

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ABSTRACT

In the present paper, we proved a coincidence and common fixed-point theorem using compatible mapping of type (P) in Intuitionistic *N*- fuzzy metric space.

Key Words: Intuitionistic *N*- fuzzy metric space, coincidence point compatible mapping, compatible mapping of type (P).

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1. Introduction

In the history of fixed-point theory, fuzzy metric spaces has wide application. In 2004 park[6] introduced intuitionistic fuzzy metric spaces ($I\ F\ M\ S_s$) as a generalization of $F\ M\ S_s$ using the concepts of intuitionistic fuzzy sets, continuous t-norm and continuous t-conorm After that. In 2022 Azam et al.[3] introduced the concept of intuitionistic fuzzy b-metric space. ($I\ F_b$MS).

On the other hand Malviya [5] defined *N*- fuzzy metric space in 2015 by using S- metric space and b- metric space.

In 2016 Ali et al.[2] given the concept of $I\ N\ F\ M\ S_s$ and proved many fixed-point theorems. By inspiring above work we proved a coincidence and common fixed-point theorem using compatible mapping of type (P) in Intuitionistic *N*- fuzzy metric space ($I\ N\ F\ M\ S_s$), which extend the result of Agrawal et al.[1] in Intuitionistic *N*- fuzzy metric space ($I\ N\ F\ M\ S_s$).

2. Preliminaries

Definition 2.1 [4] A map $\star: [0,1] \times [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t- norm if it satisfies the following conditions:

$\mathrm{T}_1{:}\star(\mu, 1, 1) = \mu,\ \star(0,0,0) = 0$

$\mathrm{T}_2{:}\star(\mu, \nu, \rho) = \star(\mu, \rho, \nu) = \star(\nu, \rho, \mu)$

$\mathrm{T}_3{:}\star$ is continuous

$\mathrm{T}_4{:}\star\left(\mu_1, \nu_{1,} \rho_1\right) \geq \star\left(\mu_2, \nu_{2,}\ \rho_{2,}\right)$ for $\mu_1 \geq \mu_2, \nu_{1,} \geq\ \nu_{2,},\ \rho_1 \geq\ \rho_2$

examples of t-norm are (1): $\mu \star \nu \star\ \rho = \mu.\nu.\rho$ and (2): $\mu \star \nu \star\ \rho = \min\{\mu, \nu, \rho\}(\mathrm{H}-\mathrm{type})$

Definition 2.2 [4] A map $\divideontimes: [0,1] \times [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t- co-norm (CTCN) if it satisfies the following conditions:

T_1:$\circledast(\mu, 0, 0) = \mu,\ \circledast(0,0,0) = 0$

T_2:$\circledast(\mu, \nu, \rho) = \circledast(\mu, \rho, \nu) = \circledast(\nu, \rho, \mu)$

T_3:$\circledast$ is continuous

T_4:$\circledast(\mu_1, \nu_{1,} \rho_1) \geq \circledast(\mu_2, \nu_{2,}\ \rho_{2,})$ for $\mu_1 \geq \mu_2, \nu_1 \geq \nu_{2,},\ \rho_1 \geq \rho_2$

$\mu \circledast \nu \circledast \rho = \max\{\mu, \nu, \rho\}$ are called a maximum CTCN.

The concept of intuitionistic N- fuzzy metric space($N\ F\ M\ S_s$), is defined as follows:

Definition 2.3[2]:- A six tuple (X, $\mathcal{M}, \mathbb{N}, \star, \circledast$) is an intuitionistic N- fuzzy metric space($I\ N\ F\ M\ S_s$), if X is an arbitrary (non-empty) set, $\star$ is a continuous t-norm, $\circledast$ is a continuous t- co-norm (CTCN), is a real number, $\mathcal{M}$and $\mathbb{N}$ are a fuzzy sets on $X^3 \times (0, \infty)$ satisfying the following conditions for all $\mu, \nu, \rho, \xi \epsilon$ X and $t, u, v > 0$:

(i) $\mathcal{M}(\mu,\ \nu,\ \rho,\ t) + \mathbb{N}(\mu,\ \nu,\ \rho,\ t) \leq 1$

(ii) $\mathcal{M}(\mu,\ \nu,\ \rho,\ t) > 0$

(iii) $\mathcal{M}(\mu,\ \nu,\ \rho,\ t) = 1$ if and only if $\mu = \nu = \rho$

(iv)$\mathcal{M}(\mu, \nu, \rho, (u + v + t)) \geq \mathcal{M}(\mu,\ \mu, \xi, u) \star \mathcal{M}(\nu, \nu, \xi, v) \star \mathcal{M}(\rho, \rho, \xi, t)$ for all $\xi \in X$.

(v) $\mathcal{M}(\mu,\ \nu,\ \rho,\ .): (0, \infty) \to (0, 1]$ is a continuous function.

(vi) $\mathbb{N}(\mu,\ \nu,\ \rho,\ t) > 0$

(vii) $\mathbb{N}(\mu,\ \nu,\ \rho,\ t) = 1$ if and only if $\mu = \nu = \rho$

(viii)$\mathbb{N}(\mu, \nu, \rho, (u + v + t)) \leq \mathbb{N}(\mu, \mu, \xi, u) \circledast \mathbb{N}(\nu, \nu, \xi, v) \circledast \mathbb{N}(\rho, \rho, \xi, t)$ for all $\xi \in X$.

(ix) $\mathbb{N}(\mu,\ \nu,\ \rho,\ .): (0, \infty) \to (0, 1]$ is a continuous function.

Here, $\mathcal{M}(\mu,\ \nu,\ \rho,\ t)$ and $\mathbb{N}(\mu,\ \nu,\ \rho,\ t)$ are membership and non-membership function of μ, ν and ρ with respect to t.

Definition 2.4. [3] Let $a \geq 1$ be a given real number. A function f: R → R will be called strictly increasing if $t < u$ implies that $f(t) \leq f(au)$ and f is called strictly decreasing if $t < u$ implies that $f(t) \geq f(au)$.

Proposition 2.5. [3] Let (X,$\mathcal{M}, \mathbb{N}, \star, \circledast$) is an intuitionistic N- fuzzy metric space($I\ N\ F\ M\ S_s$), then for all $\mu, \nu \in X$, the fuzzy set $\mathcal{M}$ and $\mathbb{N}$ are defined with respect to product such that $\mathcal{M}(\mu, \mu, \nu \star): [0, \infty) \longrightarrow [0, 1]$ is strictly increasing and $\mathbb{N}(\mu, \mu, \nu \circledast) : [0, \infty) \longrightarrow [0, 1]$ strictly decreasing.

Definition 2.6. [2] Suppose (X,$\mathcal{M}, \mathbb{N}, \star, \circledast$) is an intuitionistic N- fuzzy metric space($I\ N\ F\ M\ S_s$), then (X,$\mathcal{M}, \mathbb{N}, \star, \circledast$) is called symmetric if

$\mathcal{M}(\mu, \mu, \nu,\ t) = \mathcal{M}(\nu, \nu, \mu,\ t)$ and $\mathbb{N}(\mu, \mu, \nu,\ t) = \mathbb{N}(\nu, \nu, \mu,\ t)$.

For all $\mu, \nu \in$ X and $t > 0$.

Definition 2.7. [2] Let $(X, \mathcal{M}, \mathbb{N}, \star, \circledast)$ is intuitionistic N- fuzzy metric space$(I\ N\ F\ M\ S_s)$,

(a) A sequence $\{\mu_n\}$ in X is said to be convergent if there exists $\mu \in X$ such that $\lim_{n\to\infty} \mathcal{M}(\mu_n, \mu_n, \mu, t) = 1$ and $\lim_{n\to\infty} \mathbb{N}(\mu_n, \mu_n, \mu, t) = 0$ $\forall t > 0$. In this case μ is called the limit of the sequence $\{\mu_n\}$ and we write $\lim_{n\to\infty} \mu_n = \mu$, or $\mu_n \to \mu$.

(b) A sequence $\{\mu_n\}$ in $(X, \mathcal{M}, \mathbb{N}, \star, \circledast)$ is said to be a Cauchy sequence if for every $\epsilon \in (0, 1)$, there exists $n_0 \in N$ such that $\mathcal{M}(\mu_n, \mu_n, \mu_m, t) > 1-\epsilon$ and $\mathbb{N}(\mu_n, \mu_n, \mu_m, t) < \epsilon$, $\forall m, n \geq n_0$ and $t > 0$.

(c) The space X is said to be complete if every Cauchy sequence is convergent and it is called compact if every sequence has a convergent subsequence.

3. Main Result

We define compatible and compatible of type-P mappings in intuitionistic N- fuzzy metric space$(I\ N\ F\ M\ S_s)$,

Definition 3.1. Two self-mappings A and B of an intuitionistic N- fuzzy metric space$(N\ F\ M\ S_s)$, $(X, \mathcal{M}, \mathbb{N}, \star, \circledast)$ are called compatible if $\lim_{n\to\infty.} \mathcal{M}(AB\mu_n, AB\mu_n, BA\mu_n, t) = 1$ and $\lim_{n\to\infty.} \mathbb{N}(AB\mu_n, AB\mu_n, BA\mu_n, t) = 0$ whenever $\{\mu_n\}$ is a sequence in X such that $\lim_{n\to\infty} B\mu_n = \lim_{n\to\infty} A\mu_n = \mu$ for all some $\mu \in X$.

Definition 3.2. Two self-mappings A and B of an intuitionistic N- fuzzy metric space$(I\ N\ F\ M\ S_s)$, $(X, \mathcal{M}, \mathbb{N}, \star, \circledast)$ are called compatible of type (P) if $\lim_{n\to\infty.} \mathcal{M}(AA\mu_n, AA\mu_n, BB\mu_n, t) = 1$ and $\lim_{n\to\infty.} \mathbb{N}(AA\mu_n, AA\mu_n, BB\mu_n, t) = 0$ whenever $\{\mu_n\}$ is a sequence in X such that $\lim_{n\to\infty} B\mu_n = \lim_{n\to\infty} A\mu_n = \mu$ for all some $\mu \in X$.

Example 3.2.1. Let $X = \{\frac{1}{n} : n \epsilon N\} \cup \{0\}$ with $\star$ continuous t-norm and $\circledast$ continuous t-conorm defined by $p \star q = pq$ and $p \circledast q = \min\{1, p+q\}$ respectively, for $p, q \in [0,1]$. For each $t \in [0,\infty)$ and $\mu, \nu \in X$, define $(\mathcal{M}, \mathbb{N})$ by

$$\mathcal{M}(\mu, \nu, w, t) = \begin{cases} \frac{t}{t+|(\mu-w)-(w-\nu)|}, & if\ \ t > 0, \\ 0, & if\ \ t = 0, \end{cases} \quad \text{and}$$

$$\mathbb{N}(\mu, \nu, w, t) = \begin{cases} \frac{|(\mu-w)-(w-\nu)|}{t+|(\mu-w)-(w-\nu)|}, & if\ \ t > 0, \\ 1, & if\ \ t = 0, \end{cases}$$

Clearly $(X, \mathcal{M}, \mathbb{N}, \star, \circledast)$ is a intuitionistic N- fuzzy metric space$(I\ N\ F\ M\ S_s)$,

Define $A\mu = \frac{\mu}{6}$ and $B\mu = \frac{\mu}{2}$ on X and $\mu_n = \frac{1}{n}$.

Clearly, it can be easily observed that A and B are compatible of type (P) mapping.

Our main result extends and generalize the fixed-point theorem of Agrawal et al [1], which is the application of compatible of type (P) mapping in the new structure of intuitionistic N- fuzzy metric space$(I\ N\ F\ M\ S_s)$,

Theorem 3.3 Let $(X, \mathcal{M}, \mathbb{N}, \star, \circledast)$ be a complete intuitionistic N- fuzzy metric space$(I\ N\ F\ M\ S_s)$, with $\star$ t-norm and $\circledast$ t-conorm defined as:

(i) $\mu \star \nu = \min\{\mu, \nu\}$, $\mu \circledast \nu = \max\{\mu, \nu\}$,

(ii) $\mathcal{M}(\mu, \mu, \nu\ \star)$ and $\mathbb{N}(\mu, \mu, \nu\ \circledast)$ are strictly increasing and strictly decreasing functions, respectively.

Let $B, A: X \to X$ be two self-mapping on X satisfy following conditions:

(i) $A(X) \subseteq B(X)$

(ii) One of B or A is continuous.

(iii) (B, A) is compatible of type (P)

(iv) If for all $\mu, \nu \in X, k \in \left(0, \frac{1}{3}\right), t > 0$,

$$\mathcal{M}(A\mu, A\mu, A\nu, kt) \geq \min\{\mathcal{M}(A\mu, A\mu, B\nu, t), \mathcal{M}(A\nu, A\nu, B\nu, t), \mathcal{M}(A\nu, A\nu, B\mu, t)\}$$

$$\mathbb{N}(A\mu, A\mu, A\nu, kt) \leq \max\{\mathbb{N}(A\mu, A\mu, B\nu, t), \mathbb{N}\, A\nu, A\nu, B\nu, t), \mathbb{N}(A\nu, A\nu, B\mu, t)\}$$

Then μ is common fixed point of B and A.

Proof. Let $\mu_0 \in X$. Since $A(X) \subseteq B(X)$ there exist μ_{2n+1} and μ_{2n} in X such that

$$\mathbb{R}\mu_{2n} = \mathbb{Q}\mu_{2n+1} = \nu_{2n+1} \quad \text{for, } n = 1, 2, 3, \ldots \qquad (3.3.1)$$

Case I. Putting $\mu = \mu_{2n}$ and $\nu = \mu_{2n+1}$ in (iv) we get

$$\mathcal{M}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) = \mathcal{M}(B\mu_{2n+1}, B\mu_{2n+1}, B\mu_{2n+2}, kt) = \mathcal{M}(A\mu_{2n}, A\mu_{2n}, A\mu_{2n+1}, kt)$$

$$\geq \min\{\mathcal{M}(A\mu_{2n}, A\mu_{2n}, B\mu_{2n+1}, t), \mathcal{M}(A\mu_{2n+1}, A\mu_{2n+1}, B\mu_{2n+1}, t), \mathcal{M}(A\mu_{2n+1}, A\mu_{2n+1}, B\mu_{2n}, t)\},$$

$$= \min\{\mathcal{M}(B\mu_{2n+1}, B\mu_{2n+1}, B\mu_{2n+1}, t), \mathcal{M}(B\mu_{2n+2}, B\mu_{2n+2}, B\mu_{2n+1}, t), \mathcal{M}(B\mu_{2n+2}, B\mu_{2n+2}, B\mu_{2n}, t)\},$$

$$= \min\{\mathcal{M}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+1}, t), \mathcal{M}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), \mathcal{M}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t)\},$$

Since $\mathcal{M}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+1}, t) = 1$.

$$\mathcal{M}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \geq \min\{(1, \mathcal{M}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), \mathcal{M}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t))\},$$

$$\geq \min\{(\mathcal{M}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), \mathcal{M}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t))\},$$

Since $kt < \frac{t}{3}$ and by (ii) of theorem 3.3 $\mathcal{M}(\mu, \mu, \nu\ \star)$ is a strictly increasing function.

If $\min\{(\mathcal{M}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), \mathcal{M}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t))\}, = \mathcal{M}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t)$

Then we will reach to a contradiction $\mathcal{M}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \geq \mathcal{M}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t)$

Therefore,

$\mathcal{M}$ $(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \geq \mathcal{M}$ $(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t)$

$\geq \mathcal{M}$ $(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3}) \star \mathcal{M}$ $(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3}) \star \mathcal{M}$ $(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n}, \frac{t}{3})$ (By using (iv) of definition 2.3

$= \min \{ \mathcal{M}$ $(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3}), \mathcal{M}$ $(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n}, \frac{t}{3})\}$ (By (i) of theorem 3.3)

Since $kt < \frac{t}{3}$ and by (ii) of theorem 3.3. $\mathcal{M}(\mu, \mu, \nu \star)$ is a strictly increasing function.

If $\min \{ \mathcal{M}$ $(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+2}, \frac{t}{3}), \mathcal{M}$ $(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n}, \frac{t}{3})\} = \mathcal{M}$ $(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3})$,

Then we will again reach to contradiction,

$\mathcal{M}$ $(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \geq \mathcal{M}$ $(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, \frac{t}{3})$

Which is not possible. Therefore, $\mathcal{M}$ $(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, kt) \geq \mathcal{M}$ $(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n}, \frac{t}{3})$

In the similar manner, $\mathcal{M}$ $(\nu_{2n+3}, \nu_{2n+3}, \nu_{2n+2}, kt) \geq \mathcal{M}$ $(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3})$

In general, $\mathcal{M}$ $(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, kt) \geq \mathcal{M}$ $(\nu_n, \nu_n, \nu_{n+1}, \frac{t}{3})$ for n=1,2,3,….

And, $\mathcal{M}$ $(\nu_{n+2}, \nu_{n+2}, \nu_{n+3}, kt) \geq \mathcal{M}$ $(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \frac{t}{3})$ for n=1,2,3,….

Also, it follows that,

$\mathcal{M}$ $(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, kt) \geq \mathcal{M}$ $(\nu_n, \nu_n, \nu_{n+1}, \frac{t}{3}) \geq \mathcal{M}$ $(\nu_{n-1}, \nu_{n-1}, \nu_n, \frac{t}{(3)^2 k})$.

Continuing this, we get,

$\mathcal{M}$ $(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, kt) \geq \mathcal{M}$ $(\nu_0, \nu_0, \nu_1, \frac{t}{(3)^{n+1} k^n}) \rightarrow 0$ as $n \rightarrow \infty$.

Thus, in general, when $n \rightarrow \infty$, clearly,

$1 \geq \mathcal{M}$ $(\nu_n, \nu_n, \nu_{n+1}, kt) \geq \mathcal{M}$ $(\nu_0, \nu_0, \nu_1, \frac{t}{(3)^n k^{n-1}}) \rightarrow 1$

Thus, $\lim_{n \rightarrow \infty.} \mathcal{M}(\nu_n, \nu_n, \nu_{n+1}, kt) = 1.$

Furthermore,

$\mathbb{N}$ $(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) = \mathbb{N}$ $(B\mu_{2n+1}, B\mu_{2n+1}, B\mu_{2n+2}, kt) = \mathbb{N}$ $(A\mu_{2n}, A\mu_{2n}, A\mu_{2n+1}, kt)$

$\leq \max\{ \mathbb{N}(A\mu_{2n}, A\mu_{2n}, B\mu_{2n+1}, t), \mathbb{N}(A\mu_{2n+1}, A\mu_{2n+1}, B\mu_{2n+1}, t), \mathbb{N}(A\mu_{2n+1}, A\mu_{2n+1}, B\mu_{2n}, t)\},$

$= \max\{ \mathbb{N}(B\mu_{2n+1}, B\mu_{2n+1}, B\mu_{2n+1}, t), \mathbb{N}(B\mu_{2n+2}, B\mu_{2n+2}, B\mu_{2n+1}, t), \mathbb{N}(B\mu_{2n+2}, B\mu_{2n+2}, B\mu_{2n}, t)\},$

$= \max\{ \mathbb{N}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+1}, t), \mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), \mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t)\},$

$\Rightarrow \mathbb{N}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \leq \max\{\mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), \mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t)\}$

Since $\mathbb{N}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+1}, t) = 0.$

$\mathbb{N}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \leq \max\{(1, \mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), \mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t))\},$

$\leq \max\{(\mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), \mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t))\},$

Since $kt < \frac{t}{3}$ and by of theorem 3.3 $\mathbb{N}(\mu, \mu, \nu \star)$ is a strictly decreasing function.

If $\max\{(\mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t), \mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t))\}, = \mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t)$

Then we will reach to a contradiction $\mathbb{N}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \leq \mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, t)$ is not possible.

Therefore,
$\mathbb{N}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \leq \mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n}, t)$

$\leq \mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3}) \ast \mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3}) \ast \mathbb{N}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n}, \frac{t}{3})$

$= \max\{ \mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3}), \mathbb{N}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n}, \frac{t}{3})\}$

Since $kt < \frac{t}{3}$ and by (ii) of theorem 3.3. $\mathbb{N}(\mu, \mu, \nu \ast)$ is a strictly decreasing function.

If $\max\{ \mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+2}, \frac{t}{3}), \mathbb{N}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n}, \frac{t}{3})\} = \mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3}),$

Then we will again reach to contradiction,

$\mathbb{N}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, kt) \leq \mathbb{N}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n+2}, \frac{t}{3})$

Which is not possible. Therefore, $\mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, kt) \leq \mathbb{N}(\nu_{2n+1}, \nu_{2n+1}, \nu_{2n}, \frac{t}{3})$

In the similar manner, $\mathbb{N}(\nu_{2n+3}, \nu_{2n+3}, \nu_{2n+2}, kt) \leq \mathbb{N}(\nu_{2n+2}, \nu_{2n+2}, \nu_{2n+1}, \frac{t}{3})$

In general, $\mathbb{N}(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, kt) \leq \mathbb{N}(\nu_{n}, \nu_{n}, \nu_{n+1}, \frac{t}{3})$ for n=1,2,3,….

And, $\mathbb{N}(\nu_{n+2}, \nu_{n+2}, \nu_{n+3}, kt) \leq \mathbb{N}(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \frac{t}{3})$ for n=1,2,3,….

Also, it follows that, $\mathbb{N}(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, kt) \leq \mathbb{N}(\nu_n, \nu_n, \nu_{n+1}, \frac{t}{3}) \leq \mathbb{N}(\nu_{n-1}, \nu_{n-1}, \nu_n, \frac{t}{(3)^2 k})$.

Continuing this, we get, $\mathbb{N}(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, kt) \leq \mathbb{N}(\nu_0, \nu_0, \nu_1, \frac{t}{(3)^{n+1} k^n}) \rightarrow 0$ as $n \rightarrow \infty$.

Thus, in general, when $n \rightarrow \infty$, clearly,

$$0 \leq \mathbb{N}(\nu_n, \nu_n, \nu_{n+1}, kt) \leq \mathbb{N}(\nu_0, \nu_0, \nu_1, \frac{t}{(3)^{n+1} k^{n-1}}) \rightarrow 0$$

Therefore, $\lim_{n\rightarrow\infty.} \mathbb{N}(\nu_n, \nu_n, \nu_{n+1}, kt) = 0$.

Hence, $\mathcal{M}(\nu_n, \nu_n, \nu_{n+1}, kt) \rightarrow 1$. And $\mathbb{N}(\nu_n, \nu_n, \nu_{n+1}, kt) \rightarrow 0$ as $n \rightarrow \infty$ for any $t > 0$,

Next, we show that the sequence $\{\nu_n\}$ is a Cauchy sequence.

For each $\varepsilon > 0$ and $t > 0$, we may be chosen $n_0 \in N$ such that

$$\mathcal{M}(\nu_n, \nu_n, \nu_{n+1}, t) > 1 - \varepsilon \text{ for all } n > n_0 \text{ and } \mathbb{N}(\nu_n, \nu_n, \nu_{n+1}, t) < \varepsilon \text{ for all } n > n_0$$

For $m, n \in N$, we suppose $m \geq n$. Then we have

$$\mathcal{M}(\nu_n, \nu_n, \nu_m, t) \geq \mathcal{M}(\nu_n, \nu_n, \nu_{n+1}, \frac{t}{3}) \star \mathcal{M}(\nu_n, \nu_n, \nu_{n+1}, \frac{t}{3s}) \star \mathcal{M}(\nu_{n+1}, \nu_{n+1}, \nu_m, \frac{t}{3})$$

$$\geq \mathcal{M}(\nu_n, \nu_n, \nu_{n+1}, \frac{t}{3}) \star \mathcal{M}(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \frac{t}{(3)^2}) \star \mathcal{M}(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \frac{t}{(3)^2})$$

$$\star \mathcal{M}(\nu_{n+2}, \nu_{n+2}, \nu_m, \frac{t}{(3)^2})$$

$$\geq \mathcal{M}(\nu_n, \nu_n, \nu_{n+1}, \frac{t}{3}) \star \mathcal{M}(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \frac{t}{(3)^2}) \star \mathcal{M}(\nu_{n+2}, \nu_{n+2}, \nu_{n+1}, \frac{t}{(3)^3}) \ldots$$

$$\Rightarrow \mathcal{M}(\nu_n, \nu_n, \nu_{n+1}, t) \geq (1 - \varepsilon) \star (1 - \varepsilon) \star (1 - \varepsilon) \ldots (1 - \varepsilon)$$

$$= \min\{(1 - \varepsilon), (1 - \varepsilon), (1 - \varepsilon) \ldots (1 - \varepsilon)\} = 1 - \varepsilon$$

And

$$\mathbb{N}(\nu_n, \nu_n, \nu_m, t) \leq \mathbb{N}(\nu_n, \nu_n, \nu_{n+1}, \frac{t}{3}) \ast \mathbb{N}(\nu_n, \nu_n, \nu_{n+1}, \frac{t}{3}) \ast \mathbb{N}(\nu_{n+1}, \nu_{n+1}, \nu_m, \frac{t}{3})$$

$$\leq \mathbb{N}(\nu_n, \nu_n, \nu_{n+1}, \frac{t}{3}) \ast \mathbb{N}(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \frac{t}{(3)^2}) \ast \mathbb{N}(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \frac{t}{(3)^2}) \ast \mathbb{N}(\nu_{n+2}, \nu_{n+2}, \nu_m, \frac{t}{(3)^2})$$

$$\leq \mathbb{N}(\nu_n, \nu_n, \nu_{n+1}, \frac{t}{3}) \ast \mathbb{N}(\nu_{n+1}, \nu_{n+1}, \nu_{n+2}, \frac{t}{(3)^2}) \ast \mathbb{N}(\nu_{n+2}, \nu_{n+2}, \nu_m, \frac{t}{(3)^3}) \ldots$$

$$\Rightarrow \mathbb{N}(\nu_n, \nu_n, \nu_{n+1}, t) \leq \varepsilon \ast \varepsilon \ast \varepsilon \ast \ldots \ast \varepsilon$$

$$= \max\{\varepsilon, \varepsilon, \varepsilon, \ldots \varepsilon\} = \varepsilon$$

Hence, $\{\nu_n\}$ is a Cauchy sequence in X.

Since $(X, \mathcal{M}, \mathbb{N}, \star, \circledast, k)$ is complete. In view of completeness of the space, sequence $\{\nu_n\}$ converges to some point $u \in X$. Also, its subsequence converges to the same Point i.e., $A\mu_{2n} = B\mu_{2n} \longrightarrow \alpha$. Now, shall prove $B\alpha = \alpha$ then

$\mathcal{M}(\alpha, \alpha, B\alpha, kt) \geq \mathcal{M}(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3}) \star \mathcal{M}(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3}) \star \mathcal{M}(A\mu_{2n}, A\mu_{2n}, B\alpha, \frac{kt}{3})$,

B is continuous and A, B are Compatible type P

such that $n \to \infty$. $AA\mu_{2n} \to B\alpha$, $BB\mu_{2n} \to B\alpha$,

$\mathcal{M}(\alpha, \alpha, B\alpha, kt) \geq \mathcal{M}(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3}) \star \mathcal{M}(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3}) \star \mathcal{M}(A\mu_{2n}, A\mu_{2n}, AA\mu_{2n}, \frac{kt}{3})$,

$\geq \mathcal{M}(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3}) \star \min\{\mathcal{M}(A\mu_{2n}, A\mu_{2n}, BA\mu_{2n}, \frac{t}{3}), \mathcal{M}(AA\mu_{2n}, AA\mu_{2n}, BA\mu_{2n}, \frac{t}{3})$,

$\mathcal{M}(AA\mu_{2n}, AA\mu_{2n}, B\mu_{2n}, \frac{t}{3})\}$

Since $A\mu_{2n} = B\mu_{2n} \longrightarrow \alpha$ and B and A are compatible type (p) mapping.

Therefore, as $n \to \infty$, we get, $AA\mu_{2n} \longrightarrow B\alpha$, $BB\mu_{2n} \to B\alpha$.

$\leq \mathcal{M}(\alpha, \alpha, \alpha, \frac{kt}{3}) \star \min\{\mathcal{M}(\alpha, \alpha, B\alpha, \frac{t}{3}), \mathcal{M}(B\alpha, B\alpha, B\alpha, \frac{t}{3}), \mathcal{M}(B\alpha, B\alpha, \alpha, \frac{t}{3})\}$

$\leq \mathcal{M}(\alpha, \alpha, \alpha, \frac{kt}{3}) \star \min\{\mathcal{M}(\alpha, \alpha, B\alpha, \frac{t}{3}), \mathcal{M}(B\alpha, B\alpha, B\alpha, \frac{t}{3}), \mathcal{M}(\alpha, \alpha, B\alpha, \frac{t}{3})\}$

$\Rightarrow \mathcal{M}(\alpha, \alpha, B\alpha, kt) \geq \mathcal{M}(\alpha, \alpha, B\alpha, \frac{t}{3})$

(Since, $\mathcal{M}(\alpha, \alpha, \alpha, \frac{kt}{3}) = 1$ and $\mathcal{M}(B\alpha, B\alpha, B\alpha, \frac{t}{3}) = 1$, for all $t > 0$)

Therefore, $B\alpha = \alpha$. now we will show that $A\alpha = \alpha$.

for that let $\mu = \alpha$ and $\nu = A\mu_{2n}$ then, (iv) of theorem (3.3) becomes

$\mathcal{M}(A\alpha, A\alpha, AA\mu_{2n}, kt) \geq \min\{\mathcal{M}(A\alpha, A\alpha, BA\mu_{2n}, t), \mathcal{M}(AA\mu_{2n}, AA\mu_{2n}, BA\mu_{2n}, t), \mathcal{M}(AA\mu_{2n}, AA\mu_{2n}, B\alpha, t)\}$

Since $A\mu_{2n} = B\mu_{2n} \longrightarrow \alpha$, B is continuous and B, A are compatible type (p) such that

$AA\mu_{2n} = BB\mu_{2n} = B\alpha = \alpha$

$\mathcal{M}(A\alpha, A\alpha, \alpha, kt) \geq \min\{\mathcal{M}(A\alpha, A\alpha, B\alpha, t), \mathcal{M}(\alpha, \alpha, B\alpha, t), \mathcal{M}(\alpha, \alpha, B\alpha, t)\}$

$\mathcal{M}(A\alpha, A\alpha, \alpha, kt) \geq \min\{\mathcal{M}(A\alpha, A\alpha, \alpha, t), \mathcal{M}(\alpha, \alpha, \alpha, t), \mathcal{M}(\alpha, \alpha, \alpha, t)\}$,

Since, $\mathcal{M}(\alpha, \alpha, \alpha, t) = 1$ for all $t > 0$. Therefore, $\mathcal{M}(A\alpha, A\alpha, \alpha, kt) \geq \mathcal{M}(A\alpha, A\alpha, \alpha, t)$

Thus, $A\alpha = \alpha$. Hence, α is a fixed point of A and B.

Now, we prove $B\alpha = \alpha$ for $\mathbb{N}$

$\mathbb{N}(\alpha, \alpha, B\alpha, kt) \leq \mathbb{N}(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3}) * \mathbb{N}(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3}) * \mathbb{N}(A\mu_{2n}, A\mu_{2n}, B\alpha, \frac{kt}{3})$,

B is continuous and A, B are Compatible type P

such that $n \to \infty$. $AA\mu_{2n} \to B\alpha$, $BB\mu_{2n} \to B\alpha$,

$\mathbb{N}(\alpha, \alpha, B\alpha, kt) \leq \mathbb{N}(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3}) * \mathbb{N}(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3}) * \mathbb{N}(A\mu_{2n}, A\mu_{2n}, AA\mu_{2n}, \frac{kt}{3})$,

$\leq \mathbb{N}(\alpha, \alpha, A\mu_{2n}, \frac{kt}{3}) * \max\{\mathbb{N}(A\mu_{2n}, A\mu_{2n}, BA\mu_{2n}, \frac{t}{3}), \mathbb{N}(AA\mu_{2n}, AA\mu_{2n}, BA\mu_{2n}, \frac{t}{3})$,

$\mathbb{N}(AA\mu_{2n}, AA\mu_{2n}, B\mu_{2n}, \frac{t}{3})\}$

Since $A\mu_{2n} = B\mu_{2n} \longrightarrow \alpha$ and B and A are compatible type (p) mapping.

Therefore, as $n \to \infty$, we get, $AA\mu_{2n} \longrightarrow B\alpha$, $BB\mu_{2n} \to B\alpha$.

$\leq \mathbb{N}(\alpha, \alpha, \alpha, \frac{kt}{3s}) * \max\{\mathbb{N}(\alpha, \alpha, B\alpha, \frac{t}{3}), \mathbb{N}(B\alpha, B\alpha, B\alpha, \frac{t}{3}), \mathbb{N}(B\alpha, B\alpha, \alpha, \frac{t}{3})\}$

$\leq \mathbb{N}(\alpha, \alpha, \alpha, \frac{kt}{3s}) * \max\{\mathbb{N}(\alpha, \alpha, B\alpha, \frac{t}{3}), \mathbb{N}(B\alpha, B\alpha, B\alpha, \frac{t}{3}), \mathbb{N}(\alpha, \alpha, B\alpha, \frac{t}{3})\}$

$\Rightarrow \mathbb{N}(\alpha, \alpha, B\alpha, kt) \leq \mathbb{N}(\alpha, \alpha, B\alpha, \frac{t}{3})$

(Since, $\mathbb{N}(\alpha, \alpha, \alpha, \frac{kt}{3})=0$ and $\mathbb{N}(B\alpha, B\alpha, B\alpha, \frac{t}{3})=0$, for all $t > 0$)

Therefore, $B\alpha = \alpha$. now we will show that $A\alpha = \alpha$.

for that let $\mu = \alpha$ and $\nu = A\mu_{2n}$ then, (iv) of theorem (3.3) becomes
$\mathbb{N}(A\alpha, A\alpha, AA\mu_{2n}, kt) \leq \max\{\mathbb{N}(A\alpha, A\alpha, BA\mu_{2n}, t), \mathbb{N}(AA\mu_{2n}, AA\mu_{2n}, BA\mu_{2n}, t), \mathbb{N}(AA\mu_{2n}, AA\mu_{2n}, B\alpha, t)\}$
Since $A\mu_{2n} = B\mu_{2n} \longrightarrow \alpha$, B is continuous and B, A are compatible type (p)
such that
$AA\mu_{2n} = BB\mu_{2n} = B\alpha = \alpha$
$\mathbb{N}(A\alpha, A\alpha, \alpha, kt) \leq \max\{\mathbb{N}(A\alpha, A\alpha, B\alpha, t), \mathbb{N}(\alpha, \alpha, B\alpha, t), \mathbb{N}(\alpha, \alpha, B\alpha, t)\}$
$\mathbb{N}(A\alpha, A\alpha, \alpha, kt) \leq \max\{\mathbb{N}(A\alpha, A\alpha, \alpha, t), \mathbb{N}(\alpha, \alpha, \alpha, t), \mathbb{N}(\alpha, \alpha, \alpha, t)\}$,
Since, $\mathbb{N}(\alpha, \alpha, \alpha, t)=0$ for all $t > 0$.
Therefore, $\mathbb{N}(A\alpha, A\alpha, \alpha, kt \leq \{\mathbb{N}(A\alpha, A\alpha, \alpha, t)$

Thus, $A\alpha = \alpha$. Hence, α is a fixed point of A and B.

Uniqueness. Let α' be another coincidence and common fixed point of A and B.

then $B\alpha' = A\alpha' = \alpha'$. we get

$\mathcal{M}(A\alpha, A\alpha, A\alpha', kt) \geq \min\{\mathcal{M}(A\alpha, A\alpha, B\alpha', t), \mathcal{M}(A\alpha', A\alpha', B\alpha', t), \mathcal{M}(A\alpha', A\alpha', B\alpha, t)\}$,

$\mathcal{M}(\alpha, \alpha, \alpha', kt) \geq \min\{\mathcal{M}(\alpha, \alpha, \alpha', t), \mathcal{M}(\alpha', \alpha', \alpha', t), \mathcal{M}(\alpha', \alpha', \alpha, t)\}$,

(since $\mathcal{M}(\alpha', \alpha', \alpha', t)$ for all $t > 0$)

Therefore,

$\mathcal{M}(\alpha, \alpha, \alpha', kt) \geq \mathcal{M}(\alpha, \alpha, \alpha', t) \geq \mathcal{M}(\alpha, \alpha, \alpha', \frac{t}{k}) \geq \mathcal{M}(\alpha', \alpha', \alpha, \frac{t}{k^2}) \ldots \geq \mathcal{M}(\alpha', \alpha', \alpha, \frac{t}{k^{n-1}}) \rightarrow 1$, As $n \rightarrow \infty$.

And,

$\mathbb{N}(A\alpha, A\alpha, A\alpha', kt) \leq \max\{\mathbb{N}(A\alpha, A\alpha, B\alpha', t), \mathbb{N}(A\alpha', A\alpha', B\alpha', t), \mathbb{N}(A\alpha', A\alpha', B\alpha, t)\}$,

$\mathbb{N}(\alpha, \alpha, \alpha', kt) \leq \max\{\mathbb{N}(\alpha, \alpha, \alpha', t), \mathbb{N}(\alpha', \alpha', \alpha', t), \mathbb{N}(\alpha', \alpha', \alpha, t)\}$,

(since $\mathbb{N}(\alpha', \alpha', \alpha', t)$ for all $t > 0$)

Therefore,

$\mathbb{N}(\alpha, \alpha, \alpha', kt) \leq \mathbb{N}(\alpha, \alpha, \alpha', t) \leq \mathbb{N}(\alpha, \alpha, \alpha', \frac{t}{k}) \leq \mathbb{N}(\alpha', \alpha', \alpha, \frac{t}{k^2}) \ldots \leq \mathbb{N}(\alpha', \alpha', \alpha, \frac{t}{k^{n-1}}) \rightarrow 0$, As $n \rightarrow \infty$.

We get $\alpha = \alpha'$. Therefor, α is the Coincidence and common fixed point of self- mappings A and B.

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References

[1] Agrawal S coincidence and common fixed-point theorem using compatible mapping of type (P) on intuitionistic fuzzy b-metric spaces. South East Asian J. of Mathematics and Mathematical Sciences Vol. 20, No. 1(2024), pp. xx-yy. DOI: 10.56827/SEAJMMS.2024.2001.11

[2] Ali Mohamed A., N-Intuitionistic fuzzy metric spaces and various types of mappings, Annals of Fuzzy Mathematics and informatics, Vol. 13, No. 2, (Feb. 2017), pp.277-287.

[3] Azam, A., Kanwal, S., introduction to Intuitionistic fuzzy b-metric spaces and fixed-point results, Thai Journal of Mathematics, Vol. 20, No. 1 (2022), 141-163.

[4] Kumar S., Common Fixed-Point Theorem in Fuzzy 2-Metric Spaces, Uni.Din. Bacau. Studii Si Cercetiri Sciintifice, Serial: Mathematical, 18(2008), 111-116.

[5] Malviya N., The *N*-fuzzy metric space and mappings with application DOI:10.1515/ fascmath 2015-0019. https:// DOI:10.1515/fascmath-2015-0019

[6] Park, J.H. Intuitionistic Fuzzy Metric Spaces, chaos solitons fractals 2004, 22, 1039-1046[Google scholar] [cross Ref]