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The coincidence point theorem in partial order N- Fuzzy Metric Space

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ABSTRACT

In the present paper, we extend and generalize a coupled coincidence fixed point theorem of Wang et al. [13] in partial order N-fuzzy metric space. An example also given to illustrate the main result of this paper.

Key Words: N-fuzzy metric space, coincident point, partial order N-fuzzy metric space,

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1. INTRODUCTION

The coupled fixed-point theorem has many applications. The coupled fixed point theorem was firstly obtained by Bhaskar and lakshmikantham[1]. On the other hand, there exist many generalizations of fuzzy metric space. See [Kramoand Michalek [5]. George and Veeramani [4]. Gahler [3] S. Sharma [10] S. Kumar [6] N. Malviya [7]. Yang[12].]

In 2015 N. Malviya [7] defined N-fuzzy metric space, which is the fuzzy structure of s-metric and b-metric space. In 2014 Wang et al[15] has given new conditions on fuzzy coupled coincidence point theorem. By inspiring above work, in this paper we proved a fuzzy coupled coincidence fixed point theorem in partial ordered N-fuzzy metric space [NFMS].

2. Preliminaries

Definition 2.1 [5]**.** A map $\diamond: [0,1] \times [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following conditions:

T_1: $\diamond(\alpha, 1,1) = \alpha, \ \diamond(0,0,0) = 0$

T_2: $\diamond(\alpha, \beta, \gamma) = \diamond(\alpha, \gamma, \beta) = \diamond(\beta, \gamma, \alpha)$

T_3: $\diamond(\alpha_1, \beta_1, \gamma_1) \geq \diamond(\alpha_2, \beta_2, \gamma_2,)$ $for\ \alpha_1 \geq \alpha_2, \beta_{1,} \geq \beta_{2,}, \gamma_1 \geq \gamma_2$

examples of t-norm are (1): $\alpha \diamond \beta \diamond \gamma = \alpha.\beta.\gamma$ and (2): $\alpha \diamond \beta \diamond \gamma = min\{\alpha,\beta,\gamma\}(H - type)$

A t-norm $\diamond$ is said to be positive if $\alpha \diamond \beta > 0$ for all $\alpha, \ \beta \in (0,1]$. Obviously, $\diamond_1$ and $\diamond_2$ are positive t-norms.

The concept of N -fuzzy metric space is defined as follows:

Definition 2.2[7]:- A triplet $(X, N, \diamond)$ is an N- fuzzy metric space($N\ F\ M\ S_s$), if X is an arbitrary (non-empty) set, $\diamond$ is a continuous t-norm and N is a fuzzy set on $X^3 \times (0,\infty)$ satisfying the following conditions for all $\alpha, \beta, \gamma \epsilon\ X$ and $t, u, v\ > 0$:

$(i)\ N(\alpha,\ \beta,\ \gamma,\ t) > 0$

$(ii)\ N(\alpha,\ \beta,\ \gamma,\ t) = 1\ if\ and\ only\ if\ \alpha = \ \beta = \ \gamma$

$(iii)\ N(\alpha,\ \beta,\ \gamma,\ u+v+t) \geq N(\alpha,\ \alpha\ ,\ \xi,\ u) \diamond N(\ \beta,\ \beta,\ \ \xi,\ v) \diamond\ N(\ \gamma,\ \ \gamma,\ \xi,\ t)$ for all $\xi \in X$.

$(iv)\ N(\alpha,\ \beta,\ \gamma,\ .): (0,\infty) \to (0,1]\ is\ a\ continuous\ function.$

Example 2.1.[7] Let $X = R$ be a real line and S be an S-metric on X defined by

$S(\alpha,\beta,\gamma) = |\alpha - \gamma| + |\beta - \gamma|$

$S(\alpha,\beta,\gamma) = |\beta + \gamma - 2\alpha| + |\beta - \gamma|$

Define $poqor = pqr\ for\ every\ p,q,r \in [0,1]$ and let N be the function on

$X^3 \times (0,\infty)$ define by $N(\alpha,\ \beta,\ \gamma,\ t) = \frac{t}{t+S(\alpha,\beta,\gamma)}$ for all $\alpha,\ \beta,\ \gamma \in X\ and$ t > 0.

Then (R, N,⋄) is an N-fuzzy metric space but it is not an $Q-$ fuzzy metic space and an $M-$ fuzzy metic space because N is not symmetric.

Example 2.2.[7] Let $X = R$ be a real line and S be an S-metric as defined in above Example 2.1. Define $p \diamond q \diamond r = pqr\ for\ ever\ p,q,r \in [0,1]$ and let N be the function on $X^3 \times (0,\infty)$ defined by $N(\alpha,\ \beta,\ \gamma,\ t) = \left[exp^{\left[\frac{S(\alpha,\beta,\gamma)}{t}\right]}\right]^{-1}$ for all $\alpha,\ \beta,\ \gamma \in X\ and$ t > 0.

Then $(R,N,\diamond)$ is an $N-fuzzy\ metic\ space,$ but it is not an $Q-$ fuzzy metic space $and\ an\ M-$ fuzzy metic space because N is not symmetric.

Lemma 2.1.[7]. $N(\alpha,\ \alpha,\ \beta,\ .)$ Is nondecreasing for all $\alpha,\ \beta\ in\ X$.

Definition 2.3[7]: - A sequence $\{\alpha_n\}$ in (X,N,o) is converges to $\alpha \in X$ if $N(\alpha_n,\ \alpha_n\ \alpha, t) \to 1$ or $N(\alpha,\ \alpha, \alpha_n\ , t) \to 1$ as $n \to \infty$ for each $t > 0$.That is for each $\epsilon > 0$ and $t > 0$ there exists $n_0 \in N$ such that for all $n \geq n_0$, $N(\alpha_n, \alpha_n, \alpha, t) > 1 - \epsilon\ or\ N(\alpha, \alpha, \alpha_n, t) > 1 - \epsilon$.

Lemma 2.2.[7] Let (X,N,o) be an $N - fuzzy\ metic\ space, where\ o\ is$ minimum $t - norm\ (H - type)$. Let $\{\alpha_n\}$ be a sequence in X. If $\{\alpha_n\}$ converges to $\alpha\ and\ \{\alpha_n\}$ also converges to $\beta\ then\ \alpha = \beta$. That is the limit of $\{\alpha_n\}$ if exists is unique.

Definition 2.4[7]: - Let (X,N,o) be an $N -$ fuzzy metic space and $\{\alpha_n\}$ be a sequence in X is called Cauchy sequence, if for each $\epsilon > 0$ and $t > 0$ there exists $n_0 \in N$ such that

$N(\alpha_n, \alpha_n, \alpha_m,\ t) > 1 - \epsilon$

or

$N(\alpha_m, \alpha_m, \alpha_n,\ t) > 1 - \epsilon$ for all $n, m \geq\ n_0$.

Definition 2.5[7]: - Let (X,N,o) be an $N - fuzzy\ metic\ space$. If every Cauchy sequence in X is convergent in X, then X is called a complete $N -$ fuzzy metic space.

Lemma 2.3[7]. Let (X,N,o) be an $N - fuzzy\ metic\ space, where\ o\ is$ minimum $t - norm\ (H - type)$ and $\{\alpha_n\}$ be a sequence in X. If $\{\alpha_n\}$ converges to $\alpha,\ then\ \{\alpha_n\}$ is a Cauchy sequence.

Proposition 1.1[7]:- Let $(X,N,\diamond)$ be an N-fuzzy metric space, then for all $\alpha,\ \beta \in X$ and $t > 0$, we have $N(\alpha,\alpha,\beta,t) = N(\beta,\beta,\alpha,t)$.

Definition 2.6 ([9]) Let g and f be two self-mappings on a nonempty set $X(i.e, g, f: X \to X). If\ gz = fz$ for some $z\epsilon X. Then\ z$ is called a coincident point of g and f. The mapping g and f are said to be weakly – compatible if they commute at their coincident point, i.e., $(gf)z = (fg)z$.

Definition 2.7[10] Let $\varphi = \{\emptyset: R^+ \to R^+\}$, where $R^+ = [0, +\infty)$ and each $\emptyset \in \varphi$ satisfies the following conditions:

($\varphi - 1$) $\emptyset$ is non-decreasing.

($\varphi - 2$) $\emptyset$ is upper semicontiguous from the right.

($\varphi - 2$) $\sum_{n=0}^{\infty} \emptyset^n(t) < +\infty$ for all $t > 0$ where $\emptyset^{n+1}(t) = \emptyset(\emptyset^n)(t), n\epsilon N$.

Definition 2.8[3] Let (X, $\leq$) is the partially ordered set and G be a mapping from X to itself. A sequence $\{\alpha_n\}$ in X is said to be non-decreasing if for each $n\epsilon N$, $\alpha_n \leq \alpha_{n+1}$. A mapping $g: X \to X$ is called monotonic increasing if for all $\alpha, \beta\epsilon X$ with $\alpha \leq \beta$, $f(\alpha) \leq f(\beta)$.

Definition 2.9[6] Let (X, $\leq$) is the partially ordered set and G: $X \to X$ and f: $X \to X$ be two mappings. The mapping G is said to have the mixed f-monotone property if for all $\alpha_1, \alpha_2 \epsilon X$ with $\alpha_1 \leq \alpha_2$, $f(\alpha_1) \leq f(\alpha_2)$ implies $G(\alpha_1, \beta) \leq G(\alpha_2, \beta)$ for all $\beta \epsilon X$, and for all $\beta_1, \beta_2 \epsilon X$, $f(\beta_1) \leq f(\beta_2)$ implies $G(\alpha, \beta_1) \geq G(\alpha, \beta_2)$ for all $\alpha\epsilon X$.

Definition 2.10[1] An element $(\alpha, \beta)\epsilon\ X \times X$ is called a coupled coincidence point of the mappings $G: X \times X \to X$ and f: $X \to X$ if

$G(\alpha, \beta) = f(\alpha),\ G(\beta, \alpha) = f(\beta)$,

Here $(f\alpha, f\beta)$ is called a coupled point of coincidence.

Lemma 2.3 [13] Let $\tau: [0, 1] \to [0, 1]$ be a left continuous function and o be a continuous t-norm. assume that $\tau(\alpha) \diamond \tau(\alpha) > \alpha$ for all $\alpha\epsilon(0, 1)$. Then $\tau(1) = 1$.

3. Main Result

Theorem:3.1 Let $(X, N, \diamond)$ be an $N - fuzzy\ metic\ space$ with a continuous and positive t-norm let $\leq$ be a partial order defined on X. Let $f: (0, \infty) \to (0, \infty)$ be a function satisfying $f(t) \leq t$ for all $t > 0$ and let $\tau : [0,1] \to [0,1]$ be a left continuous and increasing function satisfying $\tau(a) \diamond \tau(a) > a$ for all $a \in (0,1)$. Let F : X × X → X and f : X → X be two mappings such that F has the mixed g-monotone property and assume that f(X) is complete. Suppose that the following conditions hold:

(i) F(X × X) ⊆ f(X),

(ii) we have

$N[F(\alpha,\beta),F(\alpha,\beta),F(\gamma,\delta),\varphi(t)] \geq \tau[N[f(\alpha),f(\alpha),f(\gamma),t] \diamond N[f(\beta),f(\beta),f(\delta),t]]$ (3.1)

for all $\alpha,\beta,\gamma,\delta \in X$, $t>0$ with $f(\alpha) \leq f(\gamma)$ and $f(\beta) \geq f(\delta)$,

(iii) If a non-decreasing sequence $\{\alpha_n\} \to \alpha$, then $\alpha_n \leq \alpha$ for all $n \in \mathbb{N} \cup \{0\}$,
(iv) If a non-increasing sequence $\{\beta_n\} \to \beta$, then $\beta_n \geq \beta$ for all $n \in \mathbb{N} \cup \{0\}$.

If there exist $\alpha_0, \beta_0 \in X$ such that $f(\alpha_0) \leq F(\alpha_0,\beta_0)$, $f(\beta_0) \geq F(\beta_0,\alpha_0)$ and

$N[f(\alpha_0),f(\alpha_0),F(\alpha_0,\beta_0),t] \diamond N[f(\beta_0),f(\beta_0),F(\beta_0,\alpha_0),t] > 0$ for all $t>0$

Then exist $\alpha^\diamond$, $\beta^\diamond \in X$ such that $f(\alpha^\diamond) = F(\alpha^\diamond,\beta^\diamond)$, $f(\beta^\diamond) = F(\beta^\diamond,\alpha^\diamond)$

Proof: let $\alpha_0, \beta_0 \in X$ such that $f(\alpha_0) \leq F(\alpha_0,\beta_0)$, $f(\beta_0) \geq F(\beta_0,\alpha_0)$

Define the sequence $\{\alpha_n\}$ and $\{\beta_n\}$ in X by

$f(\alpha_{n+1}) = F(\alpha_n,\beta_n)$ and $f(\beta_{n+1}) = F(\beta_n,\alpha_n)$ for all $n \in \mathbb{N} \cup \{0\}$.

Along the lines of proof of {17}, we see that

$f(\alpha_n) \leq f(\alpha_{n+1})$ and $f(\beta_n) \geq f(\beta_{n+1})$, for all $n \in \mathbb{N} \cup \{0\}$. (3.2)

By (3.1) and (3.2) we have

$$N[f(\alpha_1),f(\alpha_1),f(\alpha_2),t] \geq N[f(\alpha_1),f(\alpha_1),f(\alpha_2),\varphi(t)]$$
$$= N[F(\alpha_0,\beta_0),F(\alpha_0,\beta_0),F(\alpha_1,\beta_1),\varphi(t)]$$
$$\geq \tau[N\{(\alpha_0,),f(\alpha_0,),f(\alpha_1),t\} \diamond N\{f(\beta_0),f(\beta_0),f(\beta_1),t\}]$$
$$> N\{(\alpha_0,),f(\alpha_0,),f(\alpha_1),t\} \diamond N\{f(\beta_0),f(\beta_0),f(\beta_1),t\} > 0,$$

for all $t>0$ (3.3)

And

$$N[f(\beta_1),f(\beta_1),f(\beta_2),t] \geq N[f(\beta_1),f(\beta_1),f(\beta_2),\varphi(t)]$$
$$= N[F(\beta_0,\alpha_0),F(\beta_0,\alpha_0),F(\beta_1,\alpha_1),\varphi(t)]$$
$$\geq \tau[N\{(\beta_0),f(\beta_0),f(\beta_1),t\} \diamond N\{f(\alpha_0),f(\alpha_0),f(\alpha_1),t\}]$$
$$> N\{(\beta_0),f(\beta_0),f(\beta_1),t\} \diamond N\{f(\alpha_0),f(\alpha_0),f(\alpha_1),t\} > 0,$$

for all $t>0$ (3.4)

Since $\diamond$ is positive, we have

$N[f(\alpha_1),f(\alpha_1),f(\alpha_2),t] \diamond N[f(\beta_1),f(\beta_1),f(\beta_2),t] > 0$, $\forall t>0$

Repeating the process (3.3) and (3.4), we get

$N[f(\alpha_2),f(\alpha_2),f(\alpha_3),t] > 0$ and $N[f(\beta_2),f(\beta_2),f(\beta_3),t] > 0$ $\forall t>0$

And further we have

$N[f(\alpha_2), f(\alpha_2), f(\alpha_3), t] \diamond N[f(\beta_2), f(\beta_2), f(\beta_3), t]\,] > 0$ $\forall$ t >0

Continuing the above process, we get, for each n $\in \mathbb{N}$,

$N[f(\alpha_n), f(\alpha_n), f(\alpha_{n+1}), t] > 0$ $\forall$ t >0

And

$N[f(\beta_n), f(\beta_n), f(\beta_{n+1}), t] > 0$ $\forall$ t >0

Since $\diamond$ is positive, one has

$N[f(\alpha_n), f(\alpha_n), f(\alpha_{n+1}), t] \diamond N[f(\beta_n), f(\beta_n), f(\beta_{n+1}), t] > 0,$ $\forall\, n \in \mathbb{N}, \forall$t >0.

Now we prove by induction that, for each $n \in \mathbb{N}$ and $k \in \mathbb{N}$ with $k \geq n$, one has

$$N[f(\alpha_n), f(\alpha_n), f(\alpha_k), t] \diamond N[f(\beta_n), f(\beta_n), f(\beta_k), t] > 0, \quad \forall t > 0 \qquad (3.5)$$

Obviously (3.5) holds for $k = n$. Assume that (3.5) holds for some $k \in \mathbb{N}$ with $k > n$, then we have

$N[f(\alpha_n), f(\alpha_n), f(\alpha_{k+1}), t] \geq N[f(\alpha_n), f(\alpha_n), f(\alpha_k), {}^t/_3] \diamond N[f(\alpha_n), f(\alpha_n), f(\alpha_k), {}^t/_3] \diamond$
$N[f(\alpha_k), f(\alpha_k), f(\alpha_{k+1}), {}^t/_3]$

Since $N[f(\alpha_n), f(\alpha_n), f(\alpha_k), {}^t/_3\,] > 0$, $N[f(\alpha_k), f(\alpha_k), f(\alpha_{k+1}), {}^t/_3] > 0$, and $\diamond$ is positive, we have

$N[f(\alpha_n), f(\alpha_n), f(\alpha_{k+1}), t] > 0$, $\forall$t >0.

Similarly, we have

$N[f(\beta_n), f(\beta_n), f(\beta_{k+1}), t] > 0$, $\forall$t >0.

Therefore, (3.5) holds for all $k \in \mathbb{N}$ with $k \geq n$.

Now we use the method of Wang[22] to show that both $\{f(\alpha_n)\}$ and $\{f(\beta_n)\}$ are Cauchy sequences. Fix t >0. Let

$$x_n = \inf_{k \geq n} N[f(\alpha_n), f(\alpha_n), f(\alpha_k), t] \diamond N[f(\beta_n), f(\beta_n), f(\beta_k), t]$$

For $k \geq n + 1$, by (3.1) and (3.2) we have

$N[f(\alpha_{n+1}), f(\alpha_{n+1}), f(\alpha_k), t] \geq N[f(\alpha_{n+1}), f(\alpha_{n+1}), f(\alpha_k), \varphi(t)]$

$\geq \tau[N\{(\alpha_n), f(\alpha_n), f(\alpha_{k-1}), t\}$
$\diamond N\{f(\beta_n), f(\beta_n), f(\beta_{k-1}), t\}]$

Similarly,

$N[f(\beta_{n+1}), f(\beta_{n+1}), f(\beta_k), t] \geq \tau[N\{(\alpha_n), f(\alpha_n), f(\alpha_{k-1}), t\}$
$\diamond N\{f(\beta_n), f(\beta_n), f(\beta_{k-1}), t\}]$.

So, by (3.5) and the hypothesis we have

$$N[f(\alpha_{n+1}), f(\alpha_{n+1}), f(\alpha_k), t] \diamond N[f(\beta_{n+1}), f(\beta_{n+1}), f(\beta_k), t]$$

$$\geq \diamond^2 \tau[N\{(\alpha_n), f(\alpha_n), f(\alpha_{k-1}), t\} \diamond N\{f(\beta_n), f(\beta_n), f(\beta_{k-1}), t\}]$$

$$\geq N\{(\alpha_n), f(\alpha_n), f(\alpha_{k-1}), t\} \diamond N\{f(\beta_n), f(\beta_n), f(\beta_{k-1}), t\} > 0. \quad (3.6)$$

Which implies that

$$x_{n+1} \geq x_n > 0.$$

Since $\{x_n\}$ is bounded, there exists $x \in (0,1]$ such that $\lim_{n\to\infty} x_n = x$. Assume that $x < 1$.

Since τ is increasing, we have

$$\diamond^2 [\tau[N\{(\alpha_n), f(\alpha_n), f(\alpha_{k-1}), t\} \diamond N\{f(\beta_n), f(\beta_n), f(\beta_{k-1}), t\}]]$$

$$\geq \diamond^2 [\inf_{k\geq n+1} [\tau[N\{(\alpha_n), f(\alpha_n), f(\alpha_{k-1}), t\} \diamond N\{f(\beta_n), f(\beta_n), f(\beta_{k-1}), t\}]]]$$

And further

$$\inf_{k\geq n+1} \diamond^2 [\tau[N\{(\alpha_n), f(\alpha_n), f(\alpha_{k-1}), t\} \diamond N\{f(\beta_n), f(\beta_n), f(\beta_{k-1}), t\}]]$$

$$\geq \diamond^2 [\inf_{k\geq n+1} [\tau[N\{(\alpha_n), f(\alpha_n), f(\alpha_{k-1}), t\} \diamond N\{f(\beta_n), f(\beta_n), f(\beta_{k-1}), t\}]]$$

(3.7)

Form (3.6) and (3.7) it follows that

$$\inf_{k\geq n+1} N[f(\alpha_{n+1}), f(\alpha_{n+1}), f(\alpha_k), t] \diamond N[f(\beta_{n+1}), f(\beta_{n+1}), f(\beta_k), t]$$

$$\geq \diamond^2 [\tau[\inf_{k\geq n+1} [N\{(\alpha_n), f(\alpha_n), f(\alpha_{k-1}), t\} \diamond N\{f(\beta_n), f(\beta_n), f(\beta_{k-1}), t\}]],$$

i.e.,

$$x_{n+1} \geq \tau(x_n) \diamond \tau(x_n), \qquad \forall n \in \mathbb{N}$$

Since τ is left continuous, by hypothesis we get

$$x \geq \tau(x) \diamond \tau(x) > x.$$

this is a contradiction. So $x = 1$.

For any given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$1 - x_n < \varepsilon$ for all $n \geq n_0$,

Thus, for each $k \geq n \geq n_0$,

$$N[f(\alpha_n), f(\alpha_n), f(\alpha_k), t] \diamond N[f(\beta_n), f(\beta_n), f(\beta_k), t] > 1 - \varepsilon,$$

Which implies that

$\text{Min}[\, N[f(\alpha_n), f(\alpha_n), f(\alpha_k), t] \diamond N[f(\beta_n), f(\beta_n), f(\beta_k), t]] > 1 - \varepsilon.$

It follows that both $\{ f(\alpha_n)\}$ and $\{ f(\beta_n)\}$ are Cauchy sequences. Since $f(X)$ is complete, there exist $\alpha^\circ, \beta^\circ \in X$ such that $f(\alpha_n) \to f(\alpha^\circ)$ and $f(\beta_n) \to f(\beta^\circ)$ as $n \to \infty$.

By hypothesis, we have

$$f(\alpha_n) \le f(\alpha^\circ) \text{ and } f(\beta_n) \ge f(\beta^\circ), \quad k \in \mathbb{N} \tag{3.8}$$

Now, for all $t > 0$, by (3.1) and (3.8) we have

$N[F(\alpha^\circ,\beta^\circ), F(\alpha^\circ,\beta^\circ), f(\alpha^\circ), t]$

$\ge N[F(\alpha^\circ,\beta^\circ), F(\alpha^\circ,\beta^\circ), F(\alpha_n,\beta_n), {}^t/_3] \diamond N[F(\alpha^\circ,\beta^\circ), F(\alpha^\circ,\beta^\circ), F(\alpha_n,\beta_n), {}^t/_3] \diamond$
$N[F(\alpha_n,\beta_n), F(\alpha_n,\beta_n), f(\alpha^\circ), {}^t/_3]$

$\ge N[F(\alpha^\circ,\beta^\circ), F(\alpha^\circ,\beta^\circ), F(\alpha_n,\beta_n), \varphi({}^t/_3)] \diamond N[F(\alpha^\circ,\beta^\circ), F(\alpha^\circ,\beta^\circ), F(\alpha_n,\beta_n), \varphi({}^t/_3)] \diamond$
$N[F(\alpha_n,\beta_n), F(\alpha_n,\beta_n), f(\alpha^\circ), \varphi({}^t/_3)]$

$\ge \tau[N[f(\alpha^\circ), f(\alpha^\circ), f(\alpha_n), {}^t/_3] \diamond N[f(\beta^\circ), f(\beta^\circ), f(\beta_n), {}^t/_3]] \diamond$
$\tau[N[f(\alpha^\circ), f(\alpha^\circ), f(\alpha_n), {}^t/_3] \diamond N[f(\beta^\circ), f(\beta^\circ), f(\beta_n), {}^t/_3]] \diamond$
$N[\mathrm{F}(\alpha_n,\beta_n), \mathrm{F}(\alpha_n,\beta_n), f(\alpha^\circ), \varphi({}^t/_3)]$ (3.9)

Since τ is left continuous and $\diamond$ is continuous, letting $n \to \infty$ in (3.9), we get

$N[\mathrm{F}(\alpha^\circ,\beta^\circ), \mathrm{F}(\alpha^\circ,\beta^\circ), f(\alpha^\circ), t]$

$\ge \lim_{n\to\infty 1} t[\tau[N[f(\alpha^\circ), f(\alpha^\circ), f(\alpha_n), {}^t/_3] \diamond N[f(\beta^\circ), f(\beta^\circ), f(\beta_n), {}^t/_3]] \diamond$
$[\tau[N[f(\alpha^\circ), f(\alpha^\circ), f(\alpha_n), {}^t/_3] \diamond N[f(\beta^\circ), f(\beta^\circ), f(\beta_n), {}^t/_3]] \diamond$
$N[\mathrm{F}(\alpha_n,\beta_n), \mathrm{F}(\alpha_n,\beta_n), f(\alpha^\circ), \varphi({}^t/_3)]$

$= \tau(1 \diamond 1) \diamond \tau(1 \diamond 1) \diamond 1 = 1,$ $\forall t > 0.$

$= 1 \diamond 1 \diamond 1 = 1,$ $\forall t > 0$

It follows that $\mathrm{F}(\alpha^\circ,\beta^\circ) = f(\alpha^\circ)$. similarly, we can prove that $\mathrm{F}(\beta^\circ,\alpha^\circ) = f(\beta^\circ)$. This completes the proof.

If $\varphi(t) = t$ for all t >0 in theorem 3.1, we get following corollary.

Corollary 3.1 Let $(X, N, \diamond)$ be an $N - fuzzy\ metic\ space$ with a continuous and positive t-norm let $\le$ be a partial order defined on X. Let $\tau : [0,1] \to [0,1]$ be a left continuous and increasing function satisfying $\tau(a) \diamond \tau(a) > a$ for all $a \in (0,1)$. Let F : X × X → X and g : X → X be two mappings such that F has the mixed g-monotone property and assume that g(X) is complete. Suppose that the following conditions hold:

(i) F(X × X) ⊆ g(X),
(ii) we have

$N[F(\alpha,\beta), F(\alpha,\beta), F(\gamma,\delta), \varphi(t)] \ge \tau[N[f(\alpha), f(\alpha), f(\gamma), t] \diamond N[f(\beta), f(\beta), f(\delta), t]]$

for all $\alpha, \beta, \gamma, \delta \in X$, $t > 0$ with $f(\alpha) \leq f(\gamma)$ and $f(\beta) \geq f(\delta)$,

(iii) If a non-decreasing sequence $\{\alpha_n\} \rightarrow x$, then $\alpha_n \leq \alpha$ for all $n \in \mathbb{N} \cup \{0\}$,
(iv) If a non-increasing sequence $\{\beta_n\} \rightarrow \beta$, then $\beta_n \geq \beta$ for all $n \in \mathbb{N} \cup \{0\}$.

If there exist $\alpha_0, \beta_0 \in X$ such that $f(\alpha_0) \leq \mathbf{F}(\alpha_0, \beta_0)$, $f(\beta_0) \geq \mathbf{F}(\beta_0, \alpha_0)$ and

$N[f(\alpha_0), f(\alpha_0), \mathbf{F}(\alpha_0, \beta_0), t] \diamond N[f(\beta_0), f(\beta_0), \mathbf{F}(\beta_0, \alpha_0), t] > 0$ for all $t > 0$

Then exist $\alpha^\diamond, \beta^\diamond \in X$ such that $f(\alpha^\diamond) = \boldsymbol{F}(\alpha^\diamond, \beta^\diamond)$, $f(\beta^\diamond) = \mathbf{F}(\beta^\diamond, \alpha^\diamond)$

Letting $f(\alpha) = \alpha$ for all $\alpha \in X$ in Theorem 3.1 and corollary 3.1, we get the following corollaries.

Corollary 3.2 Let $(X, N, \diamond)$ be an $N - fuzzy\ metic\ space$ with a continuous and positive t-norm let $\leq$ be a partial order defined on X. Let $f: (0, \infty) \rightarrow (0, \infty)$ be a function satisfying $f(t) \leq t$ for all $t > 0$ and let $\tau : [0,1] \rightarrow [0,1]$ be a left continuous and increasing function satisfying $\tau(a) \diamond \tau(a) > a$ for all $a \in (0,1)$. Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be two mappings such that F has the mixed g-monotone property and assume that g(X) is complete. Suppose that the following conditions hold:

(i) we have

$N[F(\alpha, \beta), F(\alpha, \beta), F(\gamma, \delta), \varphi(t)] \geq \tau[N[f(\alpha), f(\alpha), f(\gamma), t] \diamond N[f(\beta), f(\beta), f(\delta), t]]$

for all $\alpha, \beta, \gamma, \delta \in X$, $t > 0$ with $f(\alpha) \leq f(\gamma)$ and $f(\beta) \geq f(\delta)$,

(ii) If a non-decreasing sequence $\{\alpha_n\} \rightarrow \alpha$ for all $n \in \mathbb{N} \cup$, then $\alpha_n \leq \alpha$ for all $n \in \mathbb{N} \cup \{0\}$,
(iii) If a non-increasing sequence $\{\beta_n\} \rightarrow \beta$, then $\beta_n \geq \beta$ for all $n \in \mathbb{N} \cup \{0\}$.

If there exist $\alpha_0, \beta_0 \in X$ such that $f(\alpha_0) \leq \mathbf{F}(\alpha_0, \beta_0)$, $f(\beta_0) \geq \mathbf{F}(\beta_0, \alpha_0)$ and

$N[f(\alpha_0), f(\alpha_0), \mathbf{F}(\alpha_0, \beta_0), t] \diamond N[f(\beta_0), f(\beta_0), \mathbf{F}(\beta_0, \alpha_0), t] > 0$ for all $t > 0$

Then exist $\alpha^\diamond, \beta^\diamond \in X$ such that $\alpha^\diamond = \boldsymbol{F}(\alpha^\diamond, \beta^\diamond)$, $\beta^\diamond = \mathbf{F}(\beta^\diamond, \alpha^\diamond)$

Corollary 3.3 Let $(X, N, \diamond)$ be an $N - fuzzy\ metic\ space$ with a positive t-norm. let $\leq$ be a partial order defined on X. Let $\tau : [0,1] \rightarrow [0,1]$ be a left continuous and increasing function satisfying $\tau(a) \diamond \tau(a) > a$ for all $a \in (0,1)$. Let $F : X \times X \rightarrow X$ and assume F has the mixed monotone property. Suppose that the following conditions hold:

(i) We have

$N[F(\alpha, \beta), F(\alpha, \beta), F(\gamma, \delta), \varphi(t)] \geq \tau[N[f(\alpha), f(\alpha), f(\gamma), t] \diamond N[f(\beta), f(\beta), f(\delta), t]]$ (3.1)

for all $\alpha, \beta, \gamma, \delta \in X$, $t > 0$ with $\alpha \leq \gamma$ and $\beta \geq \delta$,

(ii) If a non-decreasing sequence $\{\alpha_n\} \to$ x, then $\alpha_n \leq \alpha$ for all n $\in \mathbb{N} \cup \{0\}$,
(iii) If a non-increasing sequence $\{\beta_n\} \to \beta$, then $\beta_n \geq \beta$ for all n $\in \mathbb{N} \cup \{0\}$.

If there exist $\alpha_0, \beta_0 \in X$ such that $\alpha_0 \leq F(\alpha_0, \beta_0)$, $\beta_0 \geq F(\beta_0, \alpha_0)$ and

$N[f(\alpha_0), f(\alpha_0), \mathbf{F}(\alpha_0, \beta_0), t] \diamond N[f(\beta_0), f(\beta_0), \mathbf{F}(\beta_0, \alpha_0), t] > 0$ for all t >0

Then exist $\alpha^\diamond, \beta^\diamond \in X$ such that $\alpha^\diamond = F(\alpha^\diamond, \beta^\diamond)$, $\beta^\diamond = F(\beta^\diamond, \alpha^\diamond)$

Theorem 3.1 by modifying, as following.

Example 3.1 Let (X, ≤) is the partially ordered set with X=[0, 1] and the natural ordering ≤ of the real numbers as the partial ordering ≤. Define N: $X^3 \times (0, \infty)$ by

$N(\alpha, \beta, \gamma, t) = e^{-\left[\frac{[\alpha-\gamma]+|\beta-\gamma|}{t}\right]}$ for all $\alpha, \beta, \gamma \in$ X, t >0

Let $p \diamond q \diamond r = pqr$ for all p, $q, r \in [0,1]$. Then Let $(X, N, \diamond)$ be an $N-$ $fuzzy\ metic\ space$.

Let $\rho(t) = t$ for all $t > 0$ and $\tau(s) = s^{\frac{1}{3}}$ for all $s\epsilon[0, 1]$. It is easy to see that $\tau(s) \diamond \tau(s) > s$ for all $s\epsilon(0, 1)$.

Define the mappings $f: X \to X$ by

$f(\alpha) = \alpha^2, \ \forall \alpha \epsilon X$, and $F: X \to X \to X$ by

$F(\alpha, \beta) = \frac{\alpha^2-\beta^2}{3} + \frac{2}{3}, \ \alpha, \beta \in$ X.

Then $\mathbf{F}(X \times X) \subseteq f(X)$, F satisfies the mixed $f-$ monotone property. Obviously $f(X)$, is complete.

Let $\alpha_0 = 0$ and $\beta_0 = 1$, then $f(\alpha_0) \leq F(\alpha_0, \beta_0)$ and $f(\beta_0) \geq F(\beta_0, \alpha_0)$.

Moreover, $N[f(\alpha_0), f(\alpha_0), \mathbf{F}(\alpha_0, \beta_0), t] \diamond N[f(\beta_0), f(\beta_0), \mathbf{F}(\beta_0, \alpha_0), t] > 0$ for all t >0.

Next we show that for all t >0 and all $\alpha, \beta, \gamma, \delta \epsilon X$ with $f(\alpha) \leq f(\gamma)$ and $f(\beta) \geq f(\delta)$, i.e.,

$\alpha \leq \gamma$ and $\beta \geq \delta$, one has

$N[F(\alpha, \beta), F(\alpha, \beta), F(\gamma, \delta), t] \geq [N[f(\alpha), f(\alpha), f(\gamma), t] \ N[f(\beta), f(\beta), f(\delta), t]^{\frac{1}{3}}$

We prove the above inequality by a contradiction. Assume

$N[F(\alpha, \beta), F(\alpha, \beta), F(\gamma, \delta), t] < [N[f(\alpha), f(\alpha), f(\gamma), t] \ N[f(\beta), f(\beta), f(\delta), t]^{\frac{1}{3}}$

Then

$e^{-2\left[\frac{[\alpha^2-\beta^2]/3 - |\gamma^2-\delta^2|/3}{t}\right]} < e^{-2\left[\frac{[\alpha^2-\gamma^2] - |\beta^2-\delta^2|}{3t}\right]}$

$[\alpha^2 - \gamma^2] - |\beta^2 - \delta^2| > [\alpha^2 - \gamma^2] + |\beta^2 - \delta^2|$.

This is a contradiction. Thus, (3.10) holds. Therefore, all the conditions of theorem 3.1 are satisfied. Then by theorem 3.1 we conclude that there exist $\alpha^{\circ}, \beta^{\circ}$ such that $f\ (\alpha^{\circ}) = \mathbf{F}(\alpha^{\circ}, \beta^{\circ})$ and $f\ (\beta^{\circ}) = \mathbf{F}(\beta^{\circ}, \alpha^{\circ})$. It is easy to see that $(\alpha^{\circ}, \beta^{\circ}) = (\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}})$, as desired.

References

[1]. Bhashkar, TG, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. 65, 1379-1393 (2006)

[2] Gahlers S., 2-Metrische Raume and ihre Topologische Structure, Math.Nachr, 26(1963), 115-148 https://doi.org/10.1002/mana.19630260109

[3] George A., Veeramani P., On Some Results in Fuzzy Metric Spaces, Fuzzy Sets Syst., 64(1994), 395-399. https://doi.org/10.1016/0165-0114(94)90162-7

[4] Kramosil O., Michalek J., Fuzzy Metrics and Statistical Metric Spaces, Kybernetika, 11(1975), 326-334.

[5] Kumar S., Common Fixed-Point Theorem in Fuzzy 2-Metric Spaces, Uni.Din. Bacau. Studii Si Cercetiri Sciintifice, Serial: Mathematical, 18(2008), 111-116.

[6] Lakahmikantham, V, Ciri ´ c, L: Coupled fixed point theorems for nonlinear contractions in partially ordered metric ´ spaces. Nonlinear Anal. 70, 4341-4349 (2009)

[7] Malviya N., The *N*-fuzzy metric space and mappings with application DOI:10.1515/ fascmath 2015-0019. https:// DOI:10.1515/fascmath-2015-0019

[8] Rao K.P.R., Altun I., Hina Bindu S., Common Coupled Fixed-Point Theorems in Generalized Fuzzy Metric Spaces, Advances in Fuzzy Systems, 2011 Article ID 986748, 6 Pages. https://doi.org/10.1155/2011/986748

[9] Saif Ur Rehman., Some Coincidence and Common Fixed-Point Results in Fuzzy Metric Space with an Application to Differential Equations, Hindawi Journal of Function Spaces, (2021), Article ID 9411993, 15 pages. https://doi.org/10.1155/2021/9411993

[10] Sedghi, S, Altun, I, Shobe, N: Coupled fixed point theorems for contractions in fuzzy metric spaces. Nonlinear Anal. 72, 1298-1304 (2010)

[11] Sharma S., On Fuzzy Metric Space, Southeast Asian Bulletin of Mathematics,26(2002), 133-145. https://doi.org/10.1007/s100120200034

[12] Sun G., Yang K., Generalized fuzzy metric spaces with properties, Research journal of Applied Sciences, Engineering and Technology, 2(7) (2010), 673-678.

[13] Wang et al.: New conditions on fuzzy coupled coincidence fixed point theorem. Fixed Point Theory and Applications 2014, 2014:153