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On the fixed point theorem for rational inequality in extended S_b metric space

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Abstract

In this paper, we proved a fixed point theorem for rational inequality in ε –chainable symmetric extended S_b- metric space. Our result extend the theorem 3.3 of Pagey and Malviya [10] in ε – chainable symmetric extended S_b- metric space with continus maps.

AMS Subject classification: 47H10, 54D05

Key words: - ε -chain, ε -chainable symmetric extended S_b- metric space, Rational inequality and Fixed point theorem.

1- Introduction and Preliminaries

In 2012, Sedghi et al.[12] defined a generalization of G metric space which is called S- metric space. In 2016, Souayan N. et al. [14] introduced the S_b-metric space as a generalization of the b-metric space and S metric Space, and proved some fixed point results under different types of contractions in a complete $S_b$metric space. In 2017, Rohen Y et al. [11] modified the definition of $S_b$metric and prove some coupled common fixed point theorems in S_b-metric space. In 2018, Mlaki et al. [7] introduesed the notion of extended S_b- metric space which generalized the S_b-metric space.

On the other hand, as long back in 1983 Cantor defined connectedness with the help of ε - chains which have been studied extensively by many mathematicians [see Reference 1,2,3,4,5,6,8,9,10] .

Very recently Malviya. [5] definfed $\varepsilon -$ chain in symmetric extended S_b-metric space with examples and proved a fixed point theorem using contraction condition. In this paper, after introduction and preliminaries in section 2, we have proved a fixed point theorem using rational expression.

Definition 1.1[7]: Let X be a nonempty set and a function $\theta: X^3 \to [1, \infty)$. An extended S_b metric on X is a function $S_\theta : X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

1. $S_\theta(x, y, z) = 0$ if and only if $x = y = z$

2 $S_\theta(x, y, z) \leq \theta(x, y, z)\, [\, S_\theta(x, x, a) + S_\theta(y, y, a) + S_\theta(z, z, a)]$

Then the function S_θ is called extended S_b - metric and the pair (X, S_θ) is called extended S_b-metric space or S_θ metric space.

Definition 1.2[7]. Let (X, S_θ) be a extended S_b-metric space then S_θ is called symmetric if

$$S_\theta(x,x,y) = S_\theta(y,y,x)\ \forall x,y\ \in X\ (\,Symmetric\ Property\,)$$

For the definitions of convergence and Cauchy sequence in extended S_b metric space reader can refer [9]

Example 1.3[7] Let $X = [0, \frac{1}{4}]$. Define $S_\theta : X^3 \to [0,\infty)$ by $S_\theta(x,\mathrm{y},\mathrm{z}) = (\max\{x,\mathrm{y}\} - \mathrm{z})^2$

And $\theta : X^3 \to [1,\infty)$ by $\theta(x,\mathrm{y},\mathrm{z}) = \max\{x,\mathrm{y}\} + z + 1$

Thus, (X, S_θ) is a completed extended S_b-metric space.

Definition 1.4[5]:- Let $(X,\ S_\theta)$ be an extended S_b-metric space. An ε- chain is a finite succession of points $a_0, a_{1,} a_2, \ldots\ldots\ldots a_{n-1}, a_n$ in X such that $S_\theta(\,a_{i-1}\,,\, a_{i-1}\,,\, a_i) \le \varepsilon$ *for* $i =$ *1,2,….,n.* The integer n is called the length of the ε – chain.

Definition 1.5 [5]:- A space $(X\,,\, S_\theta)$ is ε -chainable (ε -connected) if every pair of points in it can be joined by an ε -chain of points in the set X and $(\,X,\, S_\theta)$ is called chainable if it is ε -chainable for each positive ε .

The space $(X,\ S_\theta)$ is called complete chainable S_θ-metric space if every Cauchy sequence converges in it.

Throughout this paper length of ε -chain between any two points x and y in X means length of shortest ε -chain between points x and y in X.

Example 1.6 [5] $X = A \cup B$ and S_θ be the metric on $X \times X \times X$ to R^+ defined by

$$S_\theta(x,\mathrm{y},\mathrm{z}) = (\max\{x,\mathrm{y}\} - \mathrm{z})^2$$

And $\theta : X^3 \to [1,\infty)$ *defined* by $\theta(\mathrm{x},\mathrm{y},\mathrm{z}) = \max\{x,\mathrm{y}\} + z + 1$

where

$A = \{0,\ 1/4\ ,\ 1/8\ \}$ *and* $B = \{\ 1/4^2\ ,\ 1/8^2,\ 1/(16)^2\}$

Then Y is ε - chainable for $\varepsilon = \frac{1}{16}$ and length of biggest ε *-chain in X is 5.*

2. Main Result:

The following result provides the application of rational inequality to find fixed point in S_b metric space for continuous mappings

Theorem 2.1: Let (X, S_θ) be a complete ε -chainable symmetric extended S_b metric space and let f, g be two continuous self maps of X *,if*

$$\epsilon = S_\theta(fx,fx,gy) \le c\left[\frac{S_\theta(y,y,fx)^2 + S_\theta(x,x,gy)^2}{S_\theta(y,y,fx) + S_\theta(x,x,gy)}\right] \ldots\ldots\ldots.\ldots(1)$$

$\forall\ x,y \in X$ where $S_\theta(y, y, fx) + S_\theta(x, x, gy) \neq 0$

$\lim\limits_{m,n\to\infty} 3Mc\ \theta(x_n, x_n, x_m)(2k) < 1$ ……………………(2)

$$and\ c < 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.\ldots\ldots (3)$$

where M *is* the length of the biggest ε-chain in space (X, S_θ) such that $M \geq 0$. Then f and g have unique common fixed point.

Proof. We define a sequence $\{x_n\}$ *as* follows

$$fx_{2n+1} = x_{2n}\ and\ gx_{2n+2} = x_{2n+1}\ for\ n = 0,1,2,3 - -$$

If $x_{2n} = x_{2n+1} = x_{2n+2}$ for some n, then we see that x_{2n} is a fixed point of f and g therefore we suppose that no two consecutive terms of sequence $\{x_n\}$ are equal. If N is the length of ε-chain between x_{2n+1} and x_{2n+2} then ε-chainability of (X, S_θ) gives.

$S_\theta(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq N\varepsilon$

$\leq M\varepsilon$ as $M \geq N$

$= MS_\theta(fx_{2n+1}, fx_{2n+1}, gx_{2n+2})$ using (1)

$\leq Mc[\frac{S_\theta(x_{2n+2}, x_{2n+2}, fx_{2n+1})^2 + S_\theta(x_{2n+1}, x_{2n+1}, gx_{2n+2})^2}{S_\theta(x_{2n+2}, x_{2n+2}, fx_{2n+1}) + S_\theta(x_{2n+1}, x_{2n+1}, gx_{2n+2})}]$

$= Mc[\frac{S_\theta(x_{2n+2}, x_{2n+2}, x_{2n})^2 + S_\theta(x_{2n+1}, x_{2n+1}, x_{2n+1})^2}{S_\theta(x_{2n+2}, x_{2n+2}, x_{2n}) + S_\theta(x_{2n+1}, x_{2n+1}, x_{2n+1})}]$

$= Mc[S_\theta(x_{2n+2}, x_{2n+2}, x_{2n})]$

$\leq Mc\theta(x_{2n+2}, x_{2n+2}, x_{2n})[2S_\theta(x_{2n+2}, x_{2n+2}, x_{2n+1}) + S_\theta(x_{2n}, x_{2n}, x_{2n+1})]$

$= Mc\theta(x_{2n+2}, x_{2n+2}, x_{2n})[2S_\theta(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S_\theta(x_{2n}, x_{2n}\, x_{2n+1})]$ by Def 1.2

$\Longrightarrow (1 - 2Mc\theta(x_{2n+2}, x_{2n+2}, x_{2n}))S_\theta(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq$
$Mc\theta(x_{2n+2}, x_{2n+2}, x_{2n})S_\theta(x_{2n}, x_{2n}\, x_{2n+1})$

$\Longrightarrow S_\theta(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq \frac{Mc\theta(x_{2n+2}, x_{2n+2}, x_{2n})}{(1 - 2Mc\theta(x_{2n+2}, x_{2n+2}, x_{2n}))}[S_\theta(x_{2n}, x_{2n}, x_{2n+1})$

In general

$S_\theta(x_n, x_n, x_{n+1}) \leq \frac{Mc\theta(x_{2n+2}, x_{2n+2}, x_{2n})}{(1 - 2Mc\theta(x_{2n+2}, x_{2n+2}, x_{2n}))}[S_\theta(x_{n-1}, x_{n-1}, x_n)$

$\Longrightarrow S_\theta(x_n, x_n, x_{n+1}) \leq k[S_\theta(x_{n-1}, x_{n-1}, x_n)$

where $k = \frac{Mc\theta(x_{2n+2}, x_{2n+2}, x_{2n})}{(1 - 2Mc\theta(x_{2n+2}, x_{2n+2}, x_{2n}))}$

$\Longrightarrow S_b(x_n, x_n, x_{n+1}) = k^n\ S_b(x_0, x_0, x_1)\ for\ n = 1,2,3 - -$

Now we prove that $\{x_n\}$ is a Cauchy sequence, if m,n are positive integers such that $n < m$ then we have.

$$\Rightarrow S_\theta(x_n, x_n, x_m) \le \theta(x_n, x_n, x_m)(2k)^n S_\theta(x_1, x_1, x_0) +$$
$$\theta(x_n, x_n, x_m)\theta(x_{n+1}, x_{n+1}, x_m)(2k)^{n+1} S_\theta(x_1, x_1, x_0)$$

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$$.+\theta(x_n, x_n, x_m)\theta(x_{n+1}, x_{n+1}, x_m)\ldots\ldots\ldots.\theta(x_{m-1}, x_{m-1}, x_m)(2k)^{m-1} S_\theta(x_1, x_1, x_0)$$

Consequently , we obtain

$$S_\theta(x_n, x_n, x_m) \le$$
$$[\theta(x_1, x_1, x_m)\theta(x_2, x_2, x_m)\ldots\ldots\ldots\theta(x_{n-1}, x_{n-1}, x_m)\theta(x_n, x_n, x_m)(2k)^n S_\theta(x_1, x_1, x_0) +$$
$$+\theta(x_1, x_1, x_m)\theta(x_2, x_2, x_m)\ldots\ldots\ldots..\theta(x_n, x_n, x_m)\theta(x_{n+1}, x_{n+1}, x_m)(2k)^{n+1} S_\theta(x_1, x_1, x_0)$$

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$$+\theta(x_1, x_1, x_m)\theta(x_2, x_2, x_m)\ldots..\theta(x_{m-2}, x_{m-2}, x_m)\theta(x_{m-1}, x_{m-1}, x_m)(2k)^{m-1} S_\theta(x_1, x_1, x_0)]$$
$$\Rightarrow S_\theta(x_n, x_n, x_m) \le S_\theta(x_1, x_1, x_0) \sum_{j=n}^{m-1}(2k)^j \prod_{i=1}^{j}\theta(x_i, x_i, x_m)\ldots\ldots\ldots\ldots(4)$$

Suppose we have a series

$$.\qquad B = \sum_{n=1}^{\infty}(2k)^n \prod_{i=1}^{n}\theta(x_i, x_i, x_m)$$

And its partial sum

$$B_n = \sum_{j=1}^{n}(2k)^j \prod_{i=1}^{j}\theta(x_i, x_i, x_m)$$

In applying the ratio test and using condition (2), the series

$$\sum_{n=1}^{\infty}(2k)^n \prod_{i=1}^{n}\theta(x_i, x_i, x_m)$$

Converges

Hence, from equation (4), for $m > n$, we have

$$S_\theta(x_n, x_n, x_m) \le S_\theta(x_1, x_1, x_0)[B_{m-1} - B_n]$$

Thus , $S_\theta(x_n, x_n, x_m) \to 0\ as\ m, n \to \infty. It\ follows\ that\ \{x_n\} is\ Cauchy\ sequence$ As X is a complete space so there exists a point x in X such that $\lim_{n\to\infty} x_n = x$

.

Existence of fixed point: Since mappings f and $g$are continuous therefore existence of fixed point follows very easily. As shown below

$$x = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} fx_{2n+1} = f\lim_{n\to\infty} x_{2n+1} = fx \ (as\ n \to \infty \ \{x_{2n+1}\} \to x)$$

Similarly

$$x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} gx_{2n+2} = g\lim_{n\to\infty} x_{2n+2} = gx \ (as\ n \to \infty \ \{x_{2n+2}\} \to x)$$

$x = y$

which shows that x is a common fixed point of $f$and g.

Uniqueness:- Let u be another fixed point of f and g

$$S_\theta(fx, fx, gu) \leq c\left[\frac{S_\theta(u,u,fx)^2 + S_\theta(x,x,gu)^2}{S_\theta(u,u,fx) + S_\theta(x,x,gu)}\right]$$

$$\Rightarrow S_\theta(fx, fx, gu) \leq c[\frac{S_\theta(u,u,x)^2 + S_\theta(x,x,u)^2}{S_\theta(u,u,x) + S_\theta(x,x,u)}]$$

$\Rightarrow S_\theta(u,u,x) \leq cS_\theta(u,u,x)$

$\Rightarrow (1-c)S_\theta(u,u,x) \leq 0$

$\Rightarrow S_\theta(u,u,x) = 0$ by 2

$\Rightarrow u = x$

This complete the proof.

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