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Some fixed point theorem for assyptotically regular maps in N_b-Fuzzy Metric space

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Abstract: - In this paper, we proved two fixed point theorems in N_b-fuzzy metric space for asymptotically regular maps. These results extend and generalize theorem 2.5 and 2.6 of Goswami et. al. [7] in the setting of N_b-fuzzy metric space.

Keywords: N_b- fuzzy metric space, fixed point theorem, asymptotically regular maps, asymptotically regular sequence.
AMS subject classifications: 47H10; 54H25

Introduction and Preliminaries

In the histry of Mathematical analysis asymptotically regularity was introdused by Browder and Petryshyn [1] in 1966 , these maps plays a great role in fixed point theory.

On the other hand, Due to wide applications many generalizations of metric spaces are exist in Mathematics.Some of important generalizations are given in[2,4,5,6,8-19]. Recently in 2022, Fernandez et. al. [3] defined N_b-fuzzy metric space, quasi N_b-fuzzy metric space and pseudo N_b-fuzzy metric spaces. They proved various theorems related to convergence of sequences and analyze topology of symmetric N_b-fuzzy metric spaces. Very recentaly Malviya N[11]. define assymptotically regular sequence and maps in N_b-fuzzy metric spaces and proved a fixed point theorem.

In this paper, we proved two fixed point theorems for generagized condition using assymptotically regular maps in N_b-fuzzy metric space.

DEFINITION 1.1[20]:-A mapping $*: [0,1] \times [0,1] \times [0,1] \rightarrow [0,1]$is called a continuous t-norm if ([0,1], $*$)is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ for $a \leq c, b \leq d$. Examples of t-norms are $a * b = \min\{a, b\}, a * b = ab$ and $a * b = \max\{a + b - 1, 0\}$

DEFINITION 1.2[3]:- A triplet (X, N_b, *, k) is an N_b-fuzzy metric space, if X is an arbitrary set, $*$ is a continuous t-norm, $k \geq 1$ is a real number and N_b is a fuzzy set on $X^3 \times (0, \infty)$satisfying the following conditions for all $x, y, z, a, \in X$ and $r, s, t > 0$

(1) $N_b(x,y,z,t) > 0$
(2) $N_b(x,y,z,t) = 1$ if and only if $x = y = z$
(3) $N_b(x,y,z,k(r+s+t)) \geq N(x,x,a,r) * N(y,y,a,s) * N(z,z,a,t)$
(4) $N_b(x,y,z,.) : (0,\infty) \to (0,1)$ is a continuous function.

For other definitions related to N_b Fuzzy metric Space reader can refer [2]

DEFINITION 1.3[20]:- A mapping $\emptyset : [0,1] \to [0,1]$ is called an altering distance function if

(i) φ is strictly decreasing and left continuous.
(ii) φ(λ) = 0 if and only if λ = 1
i.e, $\lim_{\emptyset \to 1-} \emptyset(1) = 0$.

DEFINITION 2.1[11] :-Let p and q be self mappings on a N_b- fuzzy metric space (X, N_b, *, k) and $\{x_n\}$ be a sequence in X. p is said to be asymptotically regular at a point $x_0 \in$ X if

$\left(\lim_{n\to\infty} N_b\left(p^n(x_0), p^n(x_0), p^{n+1}(x_0), t\right)\right) = 1$, ∀ t > 0.

Also the sequence $\{x_n\}$ is said to be asymptotically regular with respect to the pair (p, q) if

$\lim_{n\to\infty} N_b(p(x_n), p(x_n), q(x_n), t) = 1$, ∀ t > 0.

Main Results :

THEOREM [2.1]:- Let $p: X \to X$ be a mapping on a complete symmetric N_b fuzzy metric space (X, N_b, *, k) and ϕ is the altering distance function. If p is asymptotically regular at a point $x_0 \in$ X and p satisfies,

$$\emptyset\big(N_b(p(x),p(x),p(y),t)\big) \leq h_1 \min\{\emptyset\big(N_b(x,x,y,t)\big), \emptyset\big(N_b(p(x),p(x),x,t)\big), \emptyset\big(N_b(p(x),p(x),y,t)\big)\} + h_2 \min\{\emptyset\big(N_b(x,x,y,t)\big), \emptyset\big(N_b(p(y),p(y),y,t)\big), \emptyset\big(N_b(x,x,p(y),t)\big)\} \ldots..(1)$$

For all $x, y \in X$, $t > 0$, where $h_1, h_2 > 0$ are constants such that $h_1 + h_2 < 1$,Then p has a unique fixed point in X.

Proof. We construct a sequence $\{x_n\}$ in X by $x_{n+1} = p(x_n)$ ∀ n ∈ ℕ ∪ {0}, where $x_0 \in$ X. If there exists n with $x_n = x_{n+1}$, then x_n is a fixed point of p. Suppose that $x_n \neq x_{n+1}$for all n.

To show that $\{x_n\}$ is a Cauchy sequence.

Let m, n ∈ ℕ ∪ {0}. From (1)

$$\emptyset\big(N_b(p(x_n),p(x_n),p(x_m),t)\big) \leq h_1 \min\{\emptyset\big(N_b(x_n,x_n,x_m,t)\big), \emptyset\big(N_b(p(x_n),p(x_n),x_n,t)\big), \emptyset\big(N_b(p(x_n),p(x_n),x_m,t)\big)\} + h_2 min\{\emptyset\big(N_b(x_n,x_n,x_m,t)\big), \emptyset\big(N_b(p(x_m),p(x_m),x_m,t)\big), \emptyset(N_b(x_n,x_n,p(x_m),t)\}$$

Since, p is asymptotically regular at $x_0 \in X$, taking n, m → ∞.

$$\lim_{n,m\to\infty} \emptyset\big(N_b(p(x_n), p(x_n), p(x_m), t)\big) = 0$$

$$\Rightarrow \lim_{n,m\to\infty} N_b(p(x_n), p(x_n), p(x_m), t) = 1.$$

i.e. $\{x_n\}$ is a Cauchy sequence in (X, N_b, *, k), since X is complete, therefore $x_n \to z$(say) in X as n, → ∞.

Existence of fixed point :Using (1)

$$\emptyset\big(N_b(x_{n+1}, x_{n+1}, p(z), t)\big) = \emptyset\big(N_b(p(x_n), p(x_n), p(z), t)\big)$$

$$\leq h_1 \min\{\emptyset\big(N_b(x_n, x_n, z, t)\big), \emptyset\big(N_b(p(x_n), p(x_n), x_n, t)\big), \big(N_b(p(x_n), p(x_n), z, t)\big) + h_2 \min\{\emptyset\big(N_b(x_n, x_n, z, t)\big), \emptyset\big(N_b(p(z), p(z), z, t)\big), \emptyset\big(N_b(x_n, x_n, p(z), t)\big)\}$$

$$\Rightarrow \emptyset\big(N_b(z, z, p(z), t)\big) = 0$$

$\Rightarrow p(z) = z$, establishes that z is a fixed point for p.

Uniqueness can be shown easily.

Hence, z is the unique fixed point of p.

THEOREM [2.2] :- Let (X, N_b, *, k) be a complete symmetric $N_b$fuzzy metric space, ∅ be the altering distance function and p and q be two commutative self-mappings on X such that

$$\emptyset\big(N_b(p(x), p(x), p(y), t)\big) \leq k_1[\emptyset\big(N_b(q(x), q(x), q(y), t)\big) + k_2\,((\emptyset\big(N_b(q(x), q(x), p(y), t)\big) + \emptyset\big(N_b(q(y), q(y), p(y), t)\big)] \text{ --------------(2)}$$

Where x, y ∈ $X, t > 0$ and $k_i : \mathbb{R} \to [0,1)$ $for\ i = 1,2\ and\ 0 < k_1.k_2 < 1$.

Moreover if

(i) p and q are asymptotically regular at x_0.
(ii) p(X) ⊆q(X),
(iii) p(X) or q(X) is a complete subspace of X ,

then p and q have a unique common fixed point.

Proof. Let $x_0 \in X$. Since $p(X) \subseteq q(X)$, define a sequence $\{u_n\}$ by $u_{n+1} = p(x_n) = q(x_{n+1}), n \in \mathbb{N} \cup \{0\}$. Again since p and q are asymptotically regular at x_0,

$$\lim_{n\to\infty} \emptyset(N_b(u_n, u_n, u_{n+1}, t)) = 0 \text{ ------------- (3)}$$

To show that the sequence $\{u_n\}$ is cauchy.

Suppose, theres exists $0 < \in < 1$ and two sequence of integers $\{r_n\}$ *and* $\{s_n\}$*such that* $r_n > s_n > n$,
$N_b(u_{r_n}, u_{r_n}, u_{s_n}, t) \leq 1 - \epsilon$,
$N_b(u_{r_n-1}, u_{r_n-1}, u_{s_n-1}, t) > 1 - \in$,
$N_b(u_{r_n-1}, u_{r_n-1}, u_{s_n}, t) > 1 - \in, \forall n \in N \cup \{0\}$. ---------------- (4)

Following the technique applied in theorem 2.2 of [11] we can show that

$$\lim_{n\to\infty} N_b(u_{r_n}, u_{r_n}, u_{s_n}, t) = 1 - \in, \ t > 0 \qquad \text{-----------------(5)}$$

$$\begin{aligned}\emptyset\left(N_b\left(u_{r_n+1}, u_{r_n+1}, u_{s_n+1}, t\right)\right) &= \emptyset\left(N_b\left(p(x_{r_n}), p(x_{r_n}), p(x_{s_n}), t\right)\right)\\ &\leq k_1[\emptyset(N_b\left(q(x_{r_n}), q(x_{r_n}), q(x_{s_n})\right)\\ &+ k_2\left(\emptyset\left(N_b\left(q(x_{r_n}), q(x_{r_n}), p(x_{r_n}), t\right)\right)\right.\\ &\left.+ \emptyset\left(N_b\left(\left(q(x_{s_n}), q(x_{s_n}), p(x_{s_n}), t\right)\right)\right)\right)]\end{aligned}$$

Taking $n \to \infty$ and using (3) and (5) we have

$$\emptyset(1-\in) \leq k_1\emptyset(1-\in) < \emptyset(1-\in)$$

a contradiction. Hence $\{u_n\}$ is Cauchy sequence.

Suppose that $q(X)$ is complete, then there exist $v \in q(X)$ such that

$\lim_{n\to\infty} u_n = v$. Also, for some $z \in X$ we have $q(z) = v$.

Now,

$$\begin{aligned}\emptyset\left(N_b(p(z), p(z), u_{n+1}, t)\right) &= \emptyset\left(N_b(p(z), p(z), p(x_n), t)\right)\\ &\leq k_1\left[\emptyset\left(N_b(q(z), q(z), q(x_n), t)\right)\right.\\ &\left.+ k_2\left(\emptyset\left(N_b(p(z), p(z), q(z), t)\right) + \emptyset\left(N_b(p(x_n), p(x_n), q(x_n), t)\right)\right)\right]\end{aligned}$$

For $n \to \infty$,

$$\emptyset\left(N_b(p(z), p(z), v, t)\right) \leq k_1\left[k_2\emptyset\left(N_b(p(z), p(z), v, t)\right)\right]$$

$$\Rightarrow (1 - k_1.k_2)\emptyset\left(N_b(p(z), p(z), v, t)\right) = 0$$

$$\Rightarrow \emptyset\left(N_b(p(z), p(z), v, t)\right) = 0$$

$$\Rightarrow p(z) = v$$

Therefore $p(z) = v = q(z)$ i.e., z is the coincident point of p and q.

Next, from (2),

$$\emptyset\left(N_b\big(p(p(z)),p(p(z)),p(z),t\big)\right) \le k_1\left[\emptyset\left(N_b\big(q(p(z)),q(p(z)),q(z),t\big)\right) + k_2\left(\emptyset\left(N_b\big(p(p(z)),p(p(z)),q(p(z)),t\big)\right) + \emptyset\big(N_b(p(z),p(z),q(z),t)\big)\right)\right]$$

$$= k_1\left[\emptyset\left(N_b\big(p(q(z)),p(q(z)),q(z),t\big)\right) + k_2\left(\emptyset\left(N_b\big(p(p(z)),p(p(z)),p(q(z)),t\big)\right) + \emptyset\big(N_b(p(z),p(z),q(z),t)\big)\right)\right]$$

$(since\ pq = qp)$

$$= k_1\left[\emptyset\left(N_b\big(p(p(z)),p(p(z)),p(z),t\big)\right) + k_2\left(\emptyset\left(N_b\big(p(p(z)),p(p(z)),p(p(z)),t\big)\right) + \emptyset\big(N_b(p(z),p(z),p(z),t)\big)\right)\right]$$

$$= k_1\emptyset\left(N_b\big(p(p(z)),p(p(z)),p(z),t\big)\right)$$

$$\Rightarrow (1-k_1)\emptyset\left(N_b\big(p(p(z)),p(p(z)),p(z),t\big)\right) = 0$$

$$\Rightarrow \emptyset\left(N_b\big(p(p(z)),p(p(z)),p(z),t\big)\right) = 0$$

$$\Rightarrow p\big(p(z)\big) = p(z) = v$$

Similarly, $q\big(q(z)\big) = q(z) = v$.

Hence v is a common fixed point of p and q. If $v_1$is another common point fixed point of p and q, then

$$\emptyset\big(N_b(p(v),p(v),p(v_1),t)\big) \le k_1\left[\emptyset\big(N_b(q(v),q(v),q(v_1),t)\big) + k_2\left(\emptyset\big(N_b(p(v),p(v),q(v),t)\big) + \emptyset\big(N_b(p(v_1),p(v_1),q(v_1),t)\big)\right)\right]$$

$$\Rightarrow \emptyset\big(N_b(v,v,v_1,t)\big) \le k_1\left[\emptyset\big(N_b(v,v,v_1,t)\big) + k_2\left(\emptyset\big(N_b(v,v,v,t)\big) + \emptyset\big(N_b(v_1,v_1,v_1,t)\big)\right)\right]$$

$$\Rightarrow (1-k_1)\emptyset\big(N_b(v,v,v_1,t)\big) = 0$$

$$\Rightarrow v = v_1$$

Hence p and q have a unique common fixed point in X.

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References

1. Browder, F. E. and Petryshyn, W. V., The solution by iteration of nonlinear functional equations in Banach spaces, Bull. Amer. Math. Soc. 72 (1966), 571–575
2. Dhage B.C., Generalized metric spaces and mapping with fixed points, Bulletin of the Calcutta Mathematical Society, 84(1992), 329-336.
3. Fernandez, J., Isik Huseyin, Malviya Neeraj, Jarad Fahd, N_b-fuzzy metric spaces with topological properties and applications, AIMS Mathematics, Vol No. 8(3), (2022), 5879-5898.
4. Gahlers S., 2-metrische Raume and ihretopologische structure, Math.Nachr, 26(1963), 115-148.
5. Gahlers S., Zur geometric 2-metrische raume, Revue RoumaineMath.Pures Appl., 11(1966), 665-667
6. George A., Veeramani P., On some results in fuzzy metric spaces, Fuzzy Sets Syst., 64(1994), 395-399.
7. Goswami N., Patir B. Fixed point theorems for asymptotically regular mappings in fuzzy metric Spaces, Korean J Math 27 (2019), No.4, pp 861-877
8. Kramosil O., Michalek J., Fuzzy metrics and statistical metric spaces, Kybernetika, 11(1975), 326-334.
9. Kumar S., Common fixed point theorem in fuzzy 2-metric spaces, Uni.Din. Bacau. Studii Si CercetiriSciintifice, Serial: Mathematical, 18(2008), 111-116.
10. MalviyaN.,The N-fuzzy metric space and mappings with application.DOI:10.1515/fascmath 2015-0019.
11. 10 Malviya N. On the Asymptotically Regular Maps in N_b-Fuzzy Metric Space with Applications Shodh Prakashan Vol.3, June 2026
12. Mustafa Z., Sims B., Some remarks concerning D-metric space, International Conference on Fixed Point Theory and Applications, Valencia, Spain, (2003), 189-198.
13. Mustafa Z., Sims B., A new approach to generalized metric spaces, Journal of Nonlinear and Convex Analysis, 7(2)(2006), 289-297.
14. Rao K.P.R., Altun I., HinaBindu S., Common coupled fixed point theorems in generalized fuzzy metric spaces, Advances in Fuzzy Systems, 2011 Article ID 986748, 6 Pages

15. Sedghi S., Shobe N., A common fixed point theorem in two M-fuzzy metric space, Commun, Korean. Math. Soc., 22(4)(2007), 513-526.
16. Sedghi S., Shobe N., Aliouche A., A generalization of fixed point theorems in S-metric spaces, Mat. Vesnik, 64(3)(2012), 258-266
17. Sedghi S., Shobe N., Zhou H., A common fixed point theorem in D∗-metric spaces, Fixed Point Theory and Application, (2007), Article ID 27906, 13 pages.
18. Sharma S., On fuzzy metric space, Southeast Asian Bulletin of Mathematics, 26(2002), 133-145.
19. Singh B., Chauhan M.S., Generalized fuzzy metric spaces and fixed point theorems, Bull, Cal. Math. Soc., 89(1997), 457- 460.
20. Sun G., Yang K., Generalized fuzzy metric spaces with properties, Research journal of Applied Sciences, Engineering and Technology, 2(7)(2010), 673-678.
21. Schweizer, B., Sklar, A., Statistical metric spaces, Pacific J. Math., 10 (1960), 314–334.