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On the expansive map in extended S_b metric space with application

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Abstract
In this paper, we define expansive mappings in extended S_b- metric spaces. As an application of expansive map, we obtain a result of fixed point for a expansive map in the setting of symmetric extended S_b metric space. These results extend main results of Wang et al. [13] into the structure of extended symmetric S_b metric space.

Keywords: Fixed Point, Extended symmetric S_b metric space, Expansive mapping.

Mathematics Subject Classification: 54H25, 47H10, 54E40

1. Introduction and Preliminaries.

In the literature of Mathematical Analysis there are great role of different types maps to find fixed point. There exist many types of maps in fixed point theory like contractive maps, compatible maps, expansive maps and assymptotically regular maps etc. In 1984, Wang et al. [13] proved fixed poin theorem for expansive map.

On the other hand , there are various generalization of metric spaces like as 2- Metric space Gahler [5,6], D-metric space [3,4] , G-metric Space [8,9], S-metric Space [10], b-metric Space [1], S_b metric space [11] etc. In 2017, Rohen Y et al. [12] modified the definition of S_b metric space, which was the real generalization of b- metric space and S-metric space and studied various properties and their applications in fixed point theory. In 2016, Chouhan et al. [2] defined expansive mapping in S metric spaces, In 2018 , Mlaki et al. [7] introduesed the notion of extended S_b- metric space which generalized the S_b-metric space.

In present paper, we define expansive maps in extended symmetric S_b metric space and investigate fixed point in this structure which extend the result of Wang [13] in this setting**.**

Definition 1.1 [5,6]: Let X be a nonempty set. A generalized metric (or 2-metric) on X is a function d: $X^3 \to R^+$ that satisfies the following conditions for all $x, y, z, a \in X$.

1. $d(x, y, z) \geq 0$
2. $d(x, y, z) = 0$ if and only if $x = y = z$
3. d(x, y, z) = d(p{x, y, z}), (symmetry),where p is permutation function,
4. $d(x, y, z) \leq$ d(x,y,a) + d(x,a,z) +d(a,y,z) for all $x, y, z, a \in X$.

Then the function d is called a 2- metric and the pair (X, d) is called a 2-metric space.

Definition 1.2 [8,9]: Let X be a nonempty set. A G- metric on X is a function $G: X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

1. $G(x, y, z) = 0 \; if \; x = y = z$
2. $0 < G(x, y, z) \; for \; all \; x, y \in X \; with \; x \neq y,$
3. $G(x, x, y) \leq G(x, y, z) for \; all \; x, y, z \in X \; with \; x \neq y,$
4. $G(x, y, z) = G(x, z, y) = G(y, z, x) \ldots \ldots ....$

5. $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$.

Then the function G is called an G- metric and the pair (X,G) is called a G-metric space.

Definition 1.3 [10]: Let X be a nonempty set. An S-metric on X is a function $S: X^3 \to [0,\infty)$ that satisfies the following conditions for all $x,y,z,a \in X$.

1. $S(x,y,z) \geq 0$
2. $S(x,y,z) = 0$ if and only if $x = y = z$
3. $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$

Then the function S is called an S- metric and the pair (X,S) is called an S-metric space

Definition 1.4 [1]: Let X be a nonempty set and $s \geq 1$ a given real number. A function $d: X \times X \to R^+$ is a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:

(1) $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$,

(3) $d(x, z) \leq s\,[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space

Definition 1.5 [11]: Let X be a nonempty set and $s \geq 1$ be a given real number. An S_b metric on X is a function $S_b: X^3 \to [0,\infty)$, that satisfies the following conditions for all $x,y,z,a \in X$.

1. $S_b(x,y,z) = 0$ if and only if $x = y = z$
2. $S_b(x,y,z) = S_b(x,y,z)$ for all $x, y, \in X$
3. $S_b(x,y,z) \leq s[S_b(x,x,a) + S_b(y,y,a) + S_b(z,z,a)]$

Then the function S_b is called an S_b metric and the pair (X,S_b) is called an S_b metric space.

Remark 1. In 2017, Rohen Y et al.. [12] proved that the condition 2 of Definition 1.5 is not generally true . They have defined modification of S_b metric space as follows

Definition 1.6 [12]**:** Let X be a nonempty set and $s \geq 1$ be a given real number. An S_b metric on X is a function $S_b: X^3 \to [0,\infty)$, that satisfies the following conditions for all $x,y,z,a \in X$.

1. $S_b(x,y,z) = 0$ if and only if $x = y = z$
2. $S_b(x,y,z) \leq s[S_b(x,x,a) + S_b(y,y,a) + S_b(z,z,a)]$

Then the function S_b is called an S_b metric and the pair (X,S_b) is called an S_b metric space

Definition 1.7[7]: Let X be a nonempty set and a function $\theta: X^3 \to [1,\infty)$. An extended S_b metric on X is a function $S_\theta : X^3 \to [0,\infty)$ that satisfies the following conditions for all $x,y,z,a \in X$.

1. $S_\theta(x,y,z) = 0$ if and only if $x = y = z$

2 $S_\theta(x,y,z) \leq \theta(x,y,z)\,[S_\theta(x,x,a) + S_\theta(y,y,a) + S_\theta(z,z,a)]$

Then the function S_θ is called extended S_b - metric and the pair (X,S_θ) is called extended S_b-metric space or S_θ metric space.

Definition 1.8[7]. Let (X, S_θ) be a extended S_b-metric space then S_θ is called symmetric if

$S_\theta(x,x,y) = S_\theta(y,y,x)\ \forall x,y \in X$ (*Symmetric Property*)

For the definitions of convergence and Cauchy sequence in extended S_b metric space reader can refer [7]

Example[1.9] Let $X = [0, \frac{1}{4}]$. Define $S_\theta : X^3 \to [0, \infty)$ by $S_\theta(x, y, z) = (\max\{x, y\} - z)^2$

And $\theta : X^3 \to [1, \infty)$ by $\theta(x, y, z) = \max\{x, y\} + z + 1$

Thus, (X, S_θ) is a completed symmetric extended S_b-metric space.

2. Main Results

Expansive map: We define expansive map in extended symmetric S_b- metric space as follows

Definition 2.1: Let (X, S_θ) be extended symmetric S_b- metric space. A map $f: X \to X$ is said to be an expansive mapping if there exists a constant $L > 1$ such that $S_\theta(fx, fx, fy) \geq L S_\theta(x, x, y)$for all $x, y \in X$.

Example 2.2: Let (X, S_θ) be extended symmetric S_b- metric space. Define a self map $f: X \to X$ by $fx = \beta x$ where $\beta > 1$, for all $x \in X$. Clearly f is an expansive map in X.

Theorem 2.3. Let (X, S_θ) be a complete symmetric S_b metric space . Let f be a surjective continuous self map of X satisfying.

$$S_\theta(fx, fx, fy) \geq c S_\theta(x, x, y) \ldots \ldots \ldots . . (1)$$

for every $x, y \in X, x \neq y$ and $1 < c$

$$\lim_{m,n \to \infty} \theta(x_n, x_n, x_m)(2k) < 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Then f has a unique fixed point in X.

Proof: We define a sequence $\{x_n\}$ as follows for $n = 0,1,2,3, \ldots$

$$x_n = f x_{n+1} \; (3)$$

If $x_n = x_{n+1}$ for some n then we see that x_n is a fixed point of f . Therefore, we suppose that no two consecutive terms of sequence $\{x_n\}$ are equal.

Now we put $x = x_{n+1}$ and $y = x_{n+2}$ in (1) we get

$$S_\theta\left(fx_{n+1}, fx_{n+1}, fx_{n+2}\right) \geq c S_\theta(x_{n+1}, x_{n+1}, x_{n+2})$$

$$\Rightarrow S_\theta(x_n, x_n, x_{n+1}) \geq c S_\theta(x_{n+1}, x_{n+1}, x_{n+2})$$

$$\Rightarrow S_\theta(x_{n+1}, x_{n+1}, x_{n+2}) \leq \frac{1}{c} S_\theta(x_n, x_n, x_{n+1})$$

Similarly

$\Rightarrow S_\theta(x_n, x_n, x_{n+1}) \leq \frac{1}{c} S_\theta(x_{n-1}, x_{n-1}, x_n)$ for $n = 1,2,3, \ldots$

Let $k = \frac{1}{c}$, since $c > 1$ therefoe $k < 1$

$$\Rightarrow S_\theta(x_n, x_n, x_{n+1}) \leq k^n S_\theta(x_0, x_0, x_1)$$

Now we prove that $\{x_n\}$ is a Cauchy sequence, if m,n are positive integers such that $n < m$ then we have.

$$\Rightarrow S_\theta(x_n, x_n, x_m) \leq \theta(x_n, x_n, x_m)(2k)^n S_\theta(x_1, x_1, x_0) +$$
$$\theta(x_n, x_n, x_m)\theta(x_{n+1}, x_{n+1}, x_m)(2k)^{n+1} S_\theta(x_1, x_1, x_0)$$

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$$.+\theta(x_n, x_n, x_m)\theta(x_{n+1}, x_{n+1}, x_m) \dots\dots\dots.\theta(x_{m-1}, x_{m-1}, x_m)(2k)^{m-1} S_\theta(x_1, x_1, x_0)$$

Consequently , we obtain

$$S_\theta(x_n, x_n, x_m) \leq$$
$$[\theta(x_1, x_1, x_m)\theta(x_2, x_2, x_m) \dots\dots\dots \theta(x_{n-1}, x_{n-1}, x_m)\theta(x_n, x_n, x_m)(2k)^n S_\theta(x_1, x_1, x_0) +$$
$$+\theta(x_1, x_1, x_m)\theta(x_2, x_2, x_m) \dots\dots\dots..\theta(x_n, x_n, x_m)\theta(x_{n+1}, x_{n+1}, x_m)(2k)^{n+1} S_\theta(x_1, x_1, x_0)$$

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$$+\theta(x_1, x_1, x_m)\theta(x_2, x_2, x_m) \dots..\theta(x_{m-2}, x_{m-2}, x_m)\theta(x_{m-1}, x_{m-1}, x_m)(2k)^{m-1} S_\theta(x_1, x_1, x_0)]$$

$$\Rightarrow S_\theta(x_n, x_n, x_m) \leq S_\theta(x_1, x_1, x_0) \sum_{j=n}^{m-1}(2k)^j \prod_{i=1}^{j} \theta(x_i, x_i, x_m) \dots\dots\dots\dots(4)$$

Suppose we have a series

$$. \qquad B = \sum_{n=1}^{\infty}(2k)^n \prod_{i=1}^{n} \theta(x_i, x_i, x_m)$$

And its partial sum

$$B_n = \sum_{j=1}^{n}(2k)^j \prod_{i=1}^{j} \theta(x_i, x_i, x_m)$$

In applying the ratio test and using equation (3), the series

$$\sum_{n=1}^{\infty}(2k)^n \prod_{i=1}^{n} \theta(x_i, x_i, x_m)$$

Converges

Hence, from equation (4), for $m > n$, we have

$$S_\theta(x_n, x_n, x_m) \leq S_\theta(x_1, x_1, x_0)[B_{m-1} - B_n]$$

Thus , $S_\theta(x_n, x_n, x_m) \to 0$ *as* $m, n \to \infty$. *It follows that* $\{x_n\}$*is Cauchy sequence* As X is a complete space so there exists a point x in X such that $\lim_{n\to\infty} x_n = x$

Existence of fixed point: Since mappings are continuous therefore existence of fixed point follows very easily. As shown below

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} fx_{n+1} = f \lim_{n\to\infty} x_{n+1} = fx \; (as\; n \to \infty \; \{x_{n+1}\} \to x)$$

which shows that x is a fixed point of f.

Uniqueness: Let z be another fixed point of f , that is

$$fz = z \; (5)$$

$S_\theta(x, x, z) = S_\theta(fx, fx, fz)$

$\geq cS_\theta(x, x, z)$

$\Rightarrow S_\theta(x, x, z) \geq cS_\theta(x, x, z)[by\ 5]$

$\Rightarrow (1 - c)S_\theta(x, x, z) \geq 0$

$\Rightarrow S_\theta(x, x, z) = 0 \; [As\ c > 1]$

$\Rightarrow x = z$

This completes the proof of the Theorem 2.3.

Corollary 2.4. Let (X, S_θ) be an complete extended symmetric S_b metric space and $f: X \to X$ be a continuous surjection. Suppose that there exist a positive integer n and a real number $c > 1$ such that $S_\theta(f^n x, f^n x, f^n y) \geq cS_\theta(x, x, y)$ for all $x, y \in X$. Then f has a unique fixed point in X.

Proof: From Theorem 2.3 f^n has a unique fixed point z. But $f^n(fz) = f(f^n z) = fz$, sofz is also a fixed point of f^n. Hence $fz = z$, z is a fixed point of f. Since the fixed point of$f$ is also fixed point of f^n, the fixed point of f is unique.

Example 2.5 . Let $X = R$ and (X, S_θ) be a symmetric extended S_b- metric space as defined in example (1.9). Define a self map f on X as follows $fx = 2x$ for all $x \in X$. Clearly f is an expansive mapping. If we take $c = 4$ then condition (1) holds trivially good and 0 is the unique fixed point of the map f.

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