Siddhanta's International Journal of Physics, Chemistry, Mathematics
An International Peer-Reviewed Open-Access Annual Research Journal | ISSN: 3107-4618 | Vol. 2(1), 2026 | Impact Factor: 9.9

The $\varepsilon -$ chainable extended S_b-metric spaces and fixed point theorem

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Abstract

In present paper, we introdue the notion of ε -chain in extended S_b- metric space and provide supporting example. As an application of ε -chain we prove a new result of fixed point using contration condition in extended S_b- metric space .

AMS Subject classification: 47H10, 54D05
Key words: - ε -chain , ε -chainable extended S_b- metric space, Chainable extended S_b- metric space and Fixed point theorem.

1- Introduction and Preliminaries

In 2012, Sedghi et al.[11] defined a generalization of G metric space which is called S-metric space. In 2016, Souayan N. et al. [13]introduced the S_b-metric space as a generalization of the b-metric space and S metric Space , and proved some fixed point results under different types of contractions in a complete S_b-metric space. In 2017, Rohen Y et al. [10] modified the definition of S_b--metric and prove some coupled common fixed point theorems in S_b-metric space.In 2018 , Mlaki et al. [9] introduesed the notion of extended S_b- metric space which generalized the S_b-metric space.

On the other hand, in the history of topology ε -chain plays a very important role in defining connectedness in metric space. Firsty in 1883 Cantor defined connectedness with the help of ε -chains which have been studied extensively by many mathematicians [see Reference 1,2,3,4,5,6,7] .
After introduction and preliminaries in section-2 we have defined $\varepsilon -$ chain in extended S_b-metric space with examples and proved a fixed point theorem using contractive condition.

Definition 1.1[9]: Let X be a nonempty set and a function $\theta: X^3 \to [1, \infty)$. An extended S_b metric on X is a function $S_\theta : X^3 \to [0, \infty)$ that satisfies the following conditions for all $x, y, z, a \in X$.

1. $S_\theta(x, y, z) = 0$ if and only if $x = y = z$
2 $S_\theta(x, y, z) \leq \theta(x, y, z) [S_\theta(x, x, a) + S_\theta(y, y, a) + S_\theta(z, z, a)]$

Then the function $S_\theta$is called extended S_b - metric and the pair (X, S_θ) is called extended S_b-metric space or S_θ metric space.

Definition 1.2[9]. Let (X, S_θ) be a extended S_b-metric space then S_θ is called symmetric if

$$S_\theta(x, x, y) = S_\theta(y, y, x) \quad \forall x, y \in X \ (Symmetric\ Property)$$

For the definitions of convergence and Cauchy sequence in extended S_b metric space reader can refer [9]

Example[9] Let $X = [0, \frac{1}{4}]$. Define $S_\theta : X^3 \to [0, \infty)$ by $S_\theta(x, y, z) = (\max\{x, y\} - z)^2$

And $\theta : X^3 \to [1, \infty)$ by $\theta(x, y, z) = \max\{x, y\} + z + 1$

Thus, (X, S_θ) is a completed extended S_b-metric space.

2.Main Results :

Definition 2.1:- Let (X, S_θ) be an extended S_b-metric space. An ε- chain is a finite succession of points $a_0, a_1, a_2, \dots \dots a_{n-1}, a_n$ in X such that $S_\theta(a_{i-1}, a_{i-1}, a_i) \leq \varepsilon$ *for i = 1,2,….,n.* The integer n is called the length of the ε – chain.

Defi**nition 2.2** :- A space (X, S_θ) is ε -chainable (ε -connected) if every pair of points in it can be joined by an ε -chain of points in the set X and (X, S_θ) is called chainable if it is ε -chainable for each positive ε .

The space (X, S_θ) is called complete chainable S_θ-metric space if every Cauchy sequence converges in it.

Throughout this paper length of ε -chain between any two points x and y in X means length of shortest ε -chain between points x and y in X.

Example 2.1 $X = A \cup B$ and S_θ be the metric on $X \times X \times X$ to R^+defined by

$$S_\theta(x, y, z) = (\max\{x, y\} - z)^2$$

And $\theta : X^3 \to [1, \infty)$ *defined* by $\theta(x, y, z) = \max\{x, y\} + z + 1$

where

A = {0, 1/4 , 1/8 } and B= { $1/4^2$, $1/8^2$, $1/(16)^2$}

Then Y is ε - chainable for $\varepsilon = \frac{1}{16}$ and length of biggest ε *-chain in X is 5.*

Theorem 2.1: Let (X, S_θ) be a complete ε -chainable symmetric S_θ-metric space and let *f, g* be two self maps of X *,if*

$$\epsilon = S_\theta(fx, fx, gy) \leq a_1 S_\theta(y, y, fx) + a_2 S_\theta(x, x, y) \_\_ (1)$$

for all $x,y \in X$ where $a_1, a_2,$ are non negative reals. And

$$\lim_{m,n\to\infty} (3Ma_1\, \theta(x_n, x_n, x_m) + a_1 + a_2) < 1 \dots \dots \dots (2)$$

$$\lim_{m,n\to\infty} \theta(x_n, x_n, x_m)(2k) < 1 \quad \dots \dots \dots \dots (3)$$

where M *is* the length of the biggest ε -chain in space (X, S_θ) such that $M \geq 0$. Then f and g have unique common fixed point.

Proof. We define a sequence $\{x_n\}$ *as* follows

$$fx_{2n+1} = x_{2n} \; and \; gx_{2n+2} = x_{2n+1} \;\; for \;\; n = 0,1,2,3 --$$

If $x_{2n} = x_{2n+1} = x_{2n+2}$ for some n, then we see that x_{2n} is a fixed point of f and g therefore we suppose that no two consecutive terms of sequence $\{x_n\}$ are equal. If N is the length of ε-chain between x_{2n+1} and x_{2n+2} then ε-chainability of (X, S_θ) gives.

$S_\theta(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq N\varepsilon$

$$\leq M\varepsilon \qquad \text{as } M \geq N$$

$$= MS_\theta(fx_{2n+1}, fx_{2n+1}, gx_{2n+2}) \qquad \text{using (1)}$$

$\leq M\{ a_1 S_\theta(x_{2n+2}, x_{2n+2}, fx_{2n+1}) + a_2 S_\theta(x_{2n+1}, x_{2n+1}, x_{2n+2})\}$

$= M\{ a_1 S_\theta(x_{2n+2}, x_{2n+2}, x_{2n}) + a_2 S_\theta(x_{2n+1}, x_{2n+1}, x_{2n+2})\}$

$= M\{ a_2 S_\theta(x_{2n+1}, x_{2n+1}, x_{2n+2}) + a_1 S_\theta(x_{2n+2}, x_{2n+2}, x_{2n})\}$

(by symmetric property)

$\leq M\{ a_2 S_\theta(x_{2n+1}, x_{2n+1}, x_{2n+2}) +$

$+ a_1 \theta(x_{2n+2}, x_{2n+2}, x_{2n}) \{2S_\theta(x_{2n+2}, x_{2n+2}, x_{2n+1}) + S_\theta(x_{2n}, x_{2n}, x_{2n+1})\}$

For sake of convinence let $\theta_1 = \theta(x_{2n+2}, x_{2n+2}, x_{2n})$

$\Rightarrow [1 - M(2\theta_1 a_1 + a_2)] S_\theta(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq M\{ \theta_1 a_1) S_\theta(x_{2n}, x_{2n}, x_{2n+1})\}$

$$\Rightarrow S_\theta(x_{2n+1}, x_{2n+1}, x_{2n+2}) = \frac{M\{ \theta_1 a_1)}{[1 - M(2\theta_1 a_1 + a_2)]} S_\theta(x_{2n}, x_{2n}, x_{2n+1})$$

In general

$$\Rightarrow S_\theta(x_n, x_n, x_{n+1}) = \frac{M(\theta_1 a_1)}{[1 - M(2\theta_1 a_1 + a_2)]} S_\theta(x_{n-1}, x_{n-1}, x_n)$$

$$\Rightarrow S_\theta(x_n, x_n, x_{n+1}) = k\, S_\theta(x_{n-1}, x_{n-1}, x_n) \;\; \text{where} \;\; k = \frac{M(\theta_1 a_1)}{[1 - M(2\theta_1 a_1 + a_2)]}$$

$$\Rightarrow S_\theta(x_n, x_n, x_{n+1}) = k^n S_\theta(x_0, x_0, x_1) \;\; for \;\; n = 1,2,3 --$$

Now we prove that $\{x_n\}$ is a Cauchy sequence, if m,n are positive integers such that $n < m$ then we have.

$\Longrightarrow S_\theta(x_n, x_n, x_m) \le \theta(x_n, x_n, x_m)(2k)^n S_\theta(x_1, x_1, x_0) +$

$\theta(x_n, x_n, x_m)\theta(x_{n+1}, x_{n+1}, x_m)(2k)^{n+1} S_\theta(x_1, x_1, x_0)$

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$.+\theta(x_n, x_n, x_m)\theta(x_{n+1}, x_{n+1}, x_m) \ldots\ldots\ldots.\theta(x_{m-1}, x_{m-1}, x_m)(2k)^{m-1} S_\theta(x_1, x_1, x_0)$

Consequently , we obtain

$S_\theta(x_n, x_n, x_m) \le$

$[\theta(x_1, x_1, x_m)\theta(x_2, x_2, x_m) \ldots\ldots\ldots \theta(x_{n-1}, x_{n-1}, x_m)\theta(x_n, x_n, x_m)(2k)^n S_\theta(x_1, x_1, x_0) +$

$+\theta(x_1, x_1, x_m)\theta(x_2, x_2, x_m) \ldots\ldots\ldots..\theta(x_n, x_n, x_m)\theta(x_{n+1}, x_{n+1}, x_m)(2k)^{n+1} S_\theta(x_1, x_1, x_0)$

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$+\theta(x_1, x_1, x_m)\theta(x_2, x_2, x_m) \ldots..\theta(x_{m-2}, x_{m-2}, x_m)\theta(x_{m-1}, x_{m-1}, x_m)(2k)^{m-1} S_\theta(x_1, x_1, x_0)]$

$\Longrightarrow S_\theta(x_n, x_n, x_m) \le S_\theta(x_1, x_1, x_0) \sum_{j=n}^{m-1}(2k)^j \prod_{i=1}^{j} \theta(x_i, x_i, x_m)$............(4)

Suppose we have a series

$$B = \sum_{n=1}^{\infty}(2k)^n \prod_{i=1}^{n} \theta(x_i, x_i, x_m)$$

And its partial sum

$$B_n = \sum_{j=1}^{n}(2k)^j \prod_{i=1}^{j} \theta(x_i, x_i, x_m)$$

In applying the ratio test and using equation (3), the series

$$\sum_{n=1}^{\infty}(2k)^n \prod_{i=1}^{n} \theta(x_i, x_i, x_m)$$

Converges

Hence, from equation (4), for $m > n$, we have

$$S_\theta(x_n, x_n, x_m) \le S_\theta(x_1, x_1, x_0)[B_{m-1} - B_n]$$

Thus , $S_\theta(x_n, x_n, x_m) \to 0$ *as* $m, n \to \infty$. *It follows that* $\{x_n\}$ *is Cauchy sequence* As X is a complete space so there exists a point x in X such that $\lim_{n\to\infty} x_n = x$

Existence of fixed point*:*

Consider

$S_\theta(x,x,gx) \le \theta(x,x,gx)\{2S_\theta(x,x,x_{2n}) + S_\theta(x_{2n},x_{2n},gx)\}$ by 2 of def *1.1 and symmetric property*

$$S_\theta(x,x,gx) \le \theta(x,x,gx)\{2S_\theta(x,x,x_{2n}) + S_\theta(fx_{2n+1},fx_{2n+1},gx)\}$$

$$\le \theta(x,x,gx)\{2S_\theta(x,x,x_{2n}) + a_1S_\theta(x,x,fx_{2n+1}) + a_2S_\theta(x_{2n+1},x_{2n+1},x)\}$$

$$= \theta(x,x,gx)\{2S_\theta(x,x,x_{2n}) + a_1S_\theta(x,x,x_{2n}) + a_2S_\theta(x_{2n+1},x_{2n+1},x)\}$$

As $\{x_{2n+1}\}$ and $\{x_{2n}\}$ are subsequences of $\{x_n\}$, *as* $n \to \infty$ $x_{2n} \to x$ *and* $x_{2n+1} \to x$

Therefore

$$S_\theta(x,x,gx) \le 0$$

$$\Rightarrow S_\theta(x,x,gx) \le 0$$

$$\Rightarrow S_\theta(x,x,gx) = 0$$

$$\Rightarrow gx = x$$

Similarly we can prove fx = x. This shows that x is common fixed point of f and g.
Uniqueness: Let u be another fixed point of *f* and g

$$S_\theta(x,x,u) = S_\theta(fx,fx,gu)$$

$$\le a_1S_\theta(u,u,fx) + a_2S_\theta(x,x,u)$$

$$= a_1S_\theta(u,u,x) + a_2S_\theta(x,x,u)$$

$\le (a_1 + a_2)S_\theta(x,x,u)$ by symmetric property of S_θ

$$\Rightarrow [1-(a_1+a_2)]S_\theta(x,x,u) \le 0$$

$\Rightarrow S_\theta(x,x,u) = 0$ by ineuality 2 $(a_1 + a_2) < 1$

$$\Rightarrow x = u$$

This complete the proof

Acknowledgement

The author are thankful to .Dr. Geeta Agrawal , EX-Professor of Mathematics, Govt. M.V.M Bhopal for giving the valuable suggestions to improve the presentation of this paper.

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