The generalization of Banach contraction principal in *N*-Fuzzy Metric Space

Neeraj Malviya
Swami Vivekanand Govt. PG College (PMCOE) Raisen, MP, India.
Email : maths.neeraj@gmail.com

Abstract: In present paper, generalize Banach contraction principal is proved in *N*-Fuzzy Metric Space (*N*-FMS) . An example is also given in support of our result

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1.Introduction:

In 2015, Malviya N. [4] introduced the notion of *N*-fuzzy metric space (*N*-FMS) (which is the generalization of fuzzy metric space [3] and S-metric space [7]) and pseudo *N*-fuzzy metric space with some topological and convergence properties. In 2021, various results related to asymptotically regular maps and sequences are proved by Shikha et al. [8].

On the other hand , In 1922, the Polish mathematician Stefan Banach[2] proved a notable result of fixed point known as the "Banach Contraction Principle" (BCP) in metric space which is one of the most important results of analysis. Very recently Agrawal et al [1] proved a Banach contration principal in *N*-fuzzy metric space (*N*-FMS).

In this paper, we introduce generalize Banach contraction principle in the structure of N-Fuzzy metric space , which extend and generalize the theorem 2.6 of Rakic D. et.al. [5] and result of Agrawal et al.[1].We have also given an example to illustrate our result.

2.Preliminaries:

We recall the following related definitions of *N*-fuzzy metric space (*N*-FMS).

Definition 2.1 [6]:A map $\circ : [0,1]^3 \to [0,1]$ is called continuous t-norm if it satisfies the following conditions:

(1) $\circ(\alpha, 1,1) = \alpha,\ \circ(0,0,0) = 0$.
(2) $\circ(\alpha,\beta,\gamma) = \circ(\gamma,\alpha,\beta) = \circ(\beta,\gamma,\alpha)$.
(3) $\circ(\alpha_1, \beta_1,\gamma_1) \geq \circ(\alpha_2, \beta_2,\gamma_2)$ for $\alpha_1 \geq \alpha_2,\ \beta_1 \geq \beta_2, \gamma_1 \geq \gamma_2$.

Examples of t-norm are

(1) $\alpha \circ \beta \circ \gamma = \alpha \cdot \beta \cdot \gamma$(Product t-norm).
(2) $\alpha \circ \beta \circ \gamma = \min\{\alpha,\beta,\gamma\}$ (Minimum$t$-norm).

Definition 2.2 [4]:A triplet $(X, N,\circ)$is N-FMS if X is an arbitrary nonempty set, $\circ$ is continuous *t*-norm, and N is a fuzzy set on $X^3 \times (0,\infty)$ satisfying the following conditions for all $\alpha,\beta,\gamma,a \in X$ and $r,k,t > 0$:

(N_1) $N(\alpha,\beta,\gamma,t) > 0$,
(N_2) $N(\alpha,\beta,\gamma,t) = 1$ iff $\alpha = \beta = \gamma$,

(N_3) $N(\alpha,\beta,\gamma,\ (r+k+t)) \geq N(\alpha,\alpha,a,r) \circ N(\beta,\beta,a,k) \circ N(\gamma,\gamma,a,t)$,
(N_4) $N(\alpha,\beta,\gamma,\ \cdot)(0,\infty) \to (0,1]$ is a continuous function.

$N(\alpha,\beta,\gamma,t)$ is considered as the degree of nearness of α,β and γ with respect to t.

Proposition 2.3 [4]: Let $(X,N,\circ)$ be an N-FMSs. then we have

$$N(\alpha,\alpha,\beta,t) = N(\beta,\beta,\alpha\,,t),\ \text{for all } \alpha,\beta \in X \text{ and } t > 0. \text{(Symmetricity)}$$

Definition 2.4[4]: Let $(X,N,\circ)$ be an N-FMS. A sequence $\{\alpha_n\}$ in $(X,N,\circ)$converges to $\alpha \in X$ if $N(\alpha_n,\alpha_n,\alpha,t) \to 1$ or $N(\alpha,\alpha,\alpha_n,t) \to 1$ as $n \to \infty$ for each $t > 0$.which is written as

$$\lim_{n\to\infty} N(\alpha_n,\alpha_n,\alpha,t) \to 1 \text{ for all } t > 0$$

Or

$$\lim_{n\to\infty} N(\alpha,\alpha,\alpha_n,t) \to 1 \text{ for all } t > 0$$

That is, for $r > 0$ and $t > 0$ there exists $n_0 \in N$ such that for all $n \geq n_0$,
$N(\alpha_n,\alpha_n,\alpha,t) > 1-r$ or $N(\alpha,\alpha,\alpha_n\,,t) > 1-r$.

Lemma 2.5[4]: Let $(X,N,\circ)$ be an N-FMS, where $\circ$ is minimum t-norm. Let $\{\alpha_n\}$ be a sequence in X. if $\{\alpha_n\}$ converges to α and $\{\alpha_n\}$ also converges to β then $\alpha = \beta$. That is if the limit of $\{\alpha_n\}$ exists, it is unique.

Definition 2.6[4]: Let $(X,N,\circ)$ be an N-FMS and $\{\alpha_n\}$ be a sequence in X is called Cauchy sequence, if for each $r > 0$ and $t > 0$, there exists $n_0 \in N$ such that

$$N(\alpha_n,\alpha_n,\alpha_m,t) > 1-r,$$

Or

$$N(\alpha_m,\alpha_m,\alpha_n,t) > 1-r \text{ ,For all } n,m \geq n_0.$$

Definition 2.7[4]: Let $(X,N,\circ)$ be an N-FMS. If every Cauchy sequence $\{\alpha_n\}$ is convergent to a point $\alpha \in X$, then X is called a complete N-FMS.

Definition 2.8[4]: Let $(X,N,\circ)$ be an N-FMS. A subset A of X is said to be F-bounded if there exist $t > 0$ and $0 < r < 1$ such that

$$N(\alpha,\alpha,\beta,t) > 1-r, \text{ for all } \alpha,\beta \in A.$$

Definition 2.9[4]: Let $(X,N,\circ)$ be an N-FMS. A self map$f: X \to X$ is a fuzzy q-contraction if for all $\alpha,\beta \in X$ and for some $q \in (0,1)$, we have

$$N(f(\alpha),f(\alpha),f(\beta),qt) \geq N(\alpha,\alpha,\beta,t).$$

Lemma 2.10[4]: Let $(X,N,\circ)$ be an N-FMS, where $\circ$ is product t norm and $\{\alpha_n\}$ be a sequence in X. if $\{\alpha_n\}$ converges to α, then $\{\alpha_n\}$ is a Cauchy sequence.

Definition 2.11[4]**:**Let $(X,N,\circ)$ and $(X',N',\circ')$ be N-FMS. Then a function $T: X \to X'$ is said to be continuous at a point α we have $\{T\alpha_n\}$ is convergent to $T(\alpha)$.

Proposition 2.12[4]: Let $(X,N,\circ)$ be N-FMS and T be a fuzzy q-contraction. If any fixed point α of satisfies

$$N(\alpha,\alpha,\alpha,t) > 0,$$

Then

$$N(\alpha,\alpha,\alpha,t)=1$$

Lemma 2.13[4]: Let $(X,N,\circ)$ be a N-FMS. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in X and suppose $\alpha_n\to\alpha,\beta_n\to\beta$, as $n\to\infty$ and $N(\alpha,\alpha,\beta,t_n)\to N(\alpha,\alpha,\beta\,,t)$ as $n\to\infty$. Then $N(\alpha_n\alpha_n,\beta_n,t_n)\to N(\alpha,\alpha,\alpha,t)$ as $n\to\infty$.

Lemma 2.14[4]: Let $(X,N,\circ)$ be a N-FMS. If there exists $q\in(0,1)$ such that $N(\alpha,\alpha,\beta,t)\geq N\left(\alpha,\alpha,\beta,\frac{t}{q}\right)$ for all $\alpha,\beta\in X$, $t>0$ and $\lim_{t\to\infty}N(\alpha,\beta,\gamma,t)=1$.Then $\alpha=\beta$.

3. Main Result:

Theorem 3.1 Let $(X,N,\circ)$ be a complete N-FMS and let $:X\to X$. If for some $q\in\left(0,\frac{1}{3}\right)$,

(i) $\lim_{t\to\infty}N(\alpha,\beta,\gamma,t)=1$,

(ii) $N(T\alpha,T\alpha,T\,\beta,t)\geq\min\left\{N\left(\alpha,\alpha,\beta,\frac{t}{q}\right),N\left(T\alpha,T\alpha,\alpha,\frac{t}{q}\right),N\left(T\,\beta,T\,\beta,\beta,\frac{t}{q}\right),N\left(T\alpha,T\alpha,\beta,\frac{t}{q}\right),N\left(\alpha,\alpha,T\,\beta,\frac{3t}{q}\right)\right\}$

Where $\circ$ is minimum t-norm.Then,T has a unique fixed point in X .

Proof: Let $\alpha_0\in X$ and $\alpha_{n+1}=T\alpha_n$, $n\in N$ in (ii) of theorem 3.1 with $\alpha=\alpha_{n-1}$ and $\beta=\beta_{n-1}$ using (N_3) of definition of 2.2

$$N(\alpha_n,\alpha_n,\alpha_{n+1},t)=N(T\alpha_{n-1},T\alpha_{n-1},T\alpha_n,t)$$
$$\geq\min\left\{N\left(\alpha_{n-1},\alpha_{n-1},\alpha_n,\frac{t}{q}\right),N\left(T\alpha_{n-1},T\alpha_{n-1},\alpha_{n-1},\frac{t}{q}\right),N\left(T\alpha_n,T\alpha_n,\alpha_n,\frac{t}{q}\right),N\left(T\alpha_{n-1},T\alpha_{n-1},\alpha_n,\frac{t}{q}\right),N\left(\alpha_{n-1},\alpha_{n-1},T\alpha_n,\frac{3t}{q}\right)\right\}$$

$$=\min\left\{N\left(\alpha_{n-1},\alpha_{n-1},\alpha_n,\frac{t}{q}\right),N\left(\alpha_n,\alpha_n,\alpha_{n-1},\frac{t}{q}\right),N\left(\alpha_{n+1},\alpha_{n+1},\alpha_n,\frac{t}{q}\right),N\left(\alpha_n,\alpha_n,\alpha_n,\frac{t}{q}\right),N\left(\alpha_{n-1},\alpha_{n-1},\alpha_{n+1},\frac{3t}{q}\right)\right\}$$

$$\geq\min\left\{N\left(\alpha_{n-1},\alpha_{n-1},\alpha_n,\frac{t}{q}\right),N\left(\alpha_n,\alpha_n,\alpha_{n-1},\frac{t}{q}\right),N\left(\alpha_{n+1},\alpha_{n+1},\alpha_n,\frac{t}{q}\right),1\,,\left\{N\left(\alpha_{n-1},\alpha_{n-1},\alpha_n,\frac{t}{q}\right)\circ N\left(\alpha_{n-1},\alpha_{n-1},\alpha_n,\frac{t}{q}\right)\circ N\left(\alpha_{n+1},\alpha_{n+1},\alpha_n,\frac{t}{q}\right)\right\}\right\}$$

Since $\alpha\circ\beta\circ\gamma=\min\{\alpha,\beta,\gamma\}$

$$\geq \min\left\{N\left(\alpha_{n-1},\alpha_{n-1},\alpha_n,\frac{t}{q}\right),N\left(\alpha_n,\alpha_n,\alpha_{n-1},\frac{t}{q}\right),N\left(\alpha_{n+1},\alpha_{n+1},\alpha_n,\frac{t}{q}\right),\right.$$
$$1\,,\min\left\{N\left(\alpha_{n-1},\alpha_{n-1},\alpha_n\frac{t}{q}\right),N\left(\alpha_{n-1},\alpha_{n-1},\alpha_n,\frac{t}{q}\right),\right.$$
$$\left.\left.N\left(\alpha_{n+1},\alpha_{n+1},\alpha_n,\frac{t}{q}\right)\right\}\right\}$$

$$\geq \min\left\{N\left(\alpha_{n-1},\alpha_{n-1},\alpha_n,\frac{t}{q}\right),N\left(\alpha_{n-1},\alpha_{n-1},\alpha_n,\frac{t}{q}\right),N\left(\alpha_n,\alpha_n,\alpha_{n+1},\frac{t}{q}\right),\right.$$
$$1,$$
$$\min\left\{N\left(\alpha_{n-1},\alpha_{n-1},\alpha_n,\frac{t}{q}\right),N\left(\alpha_{n-1},\alpha_{n-1},\alpha_n,\frac{t}{q}\right),\right.$$
$$\left.\left.N\left(\alpha_n,\alpha_n,\alpha_{n+1},\frac{t}{q}\right)\right\}\right\}$$

[By symmetricity]

$$\geq \min\left\{N\left(\alpha_{n-1},\alpha_{n-1},\alpha_n,\frac{t}{q}\right),N\left(\alpha_n,\alpha_n,\alpha_{n+1},\frac{t}{q}\right),1,\ \min\left\{N\left(\alpha_{n-1},\alpha_{n-1},\alpha_n,\frac{t}{q}\right),\right.\right.$$
$$\left.\left.N\left(\alpha_n,\alpha_n,\alpha_{n+1},\frac{t}{q}\right)\right\}\right\}\ldots..\ (1)$$

$$\Rightarrow N(\alpha_n,\alpha_n,\alpha_{n+1},t)\geq \min\left\{N\left(\alpha_{n-1},\alpha_{n-1},\alpha_n,\frac{t}{q}\right),N\left(\alpha_{n+1},\alpha_{n+1},\alpha_n,\frac{t}{q}\right)\right\}$$

(By symmetricity and minimun function property)

Case-I

If $N(\alpha_n,\alpha_n,\alpha_{n+1},t)\geq N\left(\alpha_{n+1},\alpha_{n+1},\alpha_n,\frac{t}{q}\right)$

When $n\in N$, $t>0$, then by lemma 2.14 of it follows that $\alpha_n=\alpha_{n+1}$ for $n\in N$.

Case-II

$$N(\alpha_n,\alpha_n,\alpha_{n+1},t)\geq N\left(\alpha_{n-1},\alpha_{n-1},\alpha_n,\frac{t}{q}\right)$$
$$\geq N\left(\alpha_{n-2},\alpha_{n-2},\alpha_{n-1},\frac{t}{q^2}\right)$$
$$\vdots$$
$$\vdots$$
$$\geq N\left(\alpha_0,\alpha_0,\alpha_1,\frac{t}{q^n}\right)$$

Hence, $N(\alpha_n,\alpha_n,\alpha_{n+1},t)\geq N\left(\alpha_0,\alpha_0,\alpha_1,\frac{t}{q^n}\right)$.

Now, for Cauchy sequence $m,n>0$ and $n>m$

$N(\alpha_n,\alpha_n,\alpha_{n+m},t)\geq N\left(\alpha_n,\alpha_n,\alpha_{n+1},\frac{t}{3}\right)\circ N\left(\alpha_n,\alpha_n,\alpha_{n+1},\frac{t}{3}\right)\circ$
$N\left(\alpha_{n+m},\alpha_{n+m},\alpha_{n+1},\frac{t}{3}\right)$[by N_3 of definition 2.2]

$$= N\left(\alpha_n,\alpha_n,\alpha_{n+1},\frac{t}{3}\right)\circ N\left(\alpha_n,\alpha_n,\alpha_{n+1},\frac{t}{3}\right)\circ N\left(\alpha_{n+1},\alpha_{n+1},\alpha_{n+m},\frac{t}{3}\right)$$

[by symmetric property]

$$\geq N\left(\alpha_n,\alpha_n,\alpha_{n+1},\frac{t}{3}\right)\circ N\left(\alpha_n,\alpha_n,\alpha_{n+1},\frac{t}{3}\right)\circ N\left(\alpha_{n+1},\alpha_{n+1},\alpha_{n+2},\frac{t}{(3)^2}\right)$$
$$\circ\ N\left(\alpha_{n+1},\alpha_{n+1},\alpha_{n+2},\frac{t}{(3)^2}\right)\circ\ N\left(\alpha_{n+m},\alpha_{n+m},\alpha_{n+2},\frac{t}{(3)^2}\right)$$

$$= N\left(\alpha_n,\alpha_n,\alpha_{n+1},\frac{t}{3}\right)\circ N\left(\alpha_n,\alpha_n,\alpha_{n+1},\frac{t}{3}\right)\circ N\left(\alpha_{n+1},\alpha_{n+1},\alpha_{n+2},\frac{t}{(3)^2}\right)$$
$$\circ N\left(\alpha_{n+1},\alpha_{n+1},\alpha_{n+2},\frac{t}{(3)^2}\right)\circ N\left(\alpha_{n+2},\alpha_{n+2},\alpha_{n+m},\frac{t}{(3)^2}\right)$$
[By symmetric property]

$$\geq N\left(\alpha_0,\alpha_0,\alpha_1,\frac{t}{q^n(3)}\right)\circ\ N\left(\alpha_0,\alpha_0,\alpha_1,\frac{t}{q^n(3)}\right)\circ\ N\left(\alpha_0,\alpha_0,\alpha_1,\frac{t}{q^{n+1}(3)^2}\right)$$
$$\circ\ N\left(\alpha_0,\alpha_0,\alpha_1,\frac{t}{q^{n+1}(3)^2}\right)\dots\dots$$

By the condition (i) of 3.1and $q<\frac{1}{3}$, we get,
$\lim_{n\to\infty} N(\alpha_n,\alpha_n,\alpha_{m+n},t) = 1\circ 1\circ 1\circ 1\dots\dots\dots = 1$.
Hence,$\{\alpha_n\}$ in Cauchy sequence. Since, $(X,N,\ \circ)$ is complete N-FMS, there exists $\alpha\in X$ such that

$$\lim_{n\to\infty}\alpha_n=\alpha$$

Existence of fixed point :-

Now, we will show α is a fixed point of T .

$$N(T\alpha,T\alpha,\alpha,t)\geq N\left(T\alpha,T\alpha,T\alpha_n,\frac{t}{3}\right)\circ N\left(T\alpha,T\alpha,T\alpha_n,\frac{t}{3}\right)\circ N\left(\alpha,\alpha,T\alpha_n,\frac{t}{3}\right)$$
[By N_3 of definition of 2.2]

$$\geq \min\left\{N\left(\alpha,\alpha,\alpha_n,\frac{t}{3q}\right),N\left(T\alpha,T\alpha,\alpha,\frac{t}{3q}\right),N\left(T\alpha_n,T\alpha_n,\alpha_n,\frac{t}{3q}\right),\right.$$
$$\left.N\left(T\alpha,T\alpha,\alpha_n,\frac{t}{3q}\right),N\left(\alpha,\alpha,T\alpha_n,\frac{t}{q}\right)\right\}$$
$$\circ \min\left\{N\left(\alpha,\alpha,\alpha_n,\frac{t}{3q}\right),N\left(T\alpha,T\alpha,\alpha,\frac{t}{3q}\right),N\left(T\alpha_n,T\alpha_n,\alpha_n,\frac{t}{3q}\right),\right.$$
$$\left.N\left(T\alpha,T\alpha,\alpha_n,\frac{t}{3q}\right),N\left(\alpha,\alpha,T\alpha_n,\frac{t}{q}\right)\right\}$$
$$\circ N\left(\alpha,\alpha,T\alpha_n,\frac{t}{3}\right)$$
[By (ii) of theorem 3.1]

$$\geq \min\left\{N\left(\alpha,\alpha,\alpha_n,\frac{t}{3q}\right),N\left(T\alpha,T\alpha,\alpha,\frac{t}{3q}\right),N\left(\alpha_{n+1},\alpha_{n+1},\alpha_n,\frac{t}{3q}\right),\right.$$
$$\left.N\left(T\alpha,T\alpha,\alpha_n,\frac{t}{3q}\right),N\left(\alpha,\alpha,\alpha_{n+1},\frac{t}{q}\right)\right\}$$
$$\circ \min\left\{N\left(\alpha,\alpha,\alpha_n,\frac{t}{3q}\right),N\left(T\alpha,T\alpha,\alpha,\frac{t}{3q}\right),N\left(\alpha_{n+1},\alpha_{n+1},\alpha_n,\frac{t}{3q}\right),\right.$$
$$\left.N\left(T\alpha,T\alpha,\alpha_n,\frac{t}{3q}\right),\ N\left(\alpha,\alpha,\alpha_{n+1},\frac{t}{q}\right)\right\}\circ N\left(\alpha,\alpha,\alpha_{n+1},\frac{t}{3}\right)$$

Therefore, for all $n \in N$ and $t > 0$. Taking $n \to \infty$

$$N(T\alpha, T\alpha, \alpha, t) \geq \min\left\{1, N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), 1, \quad N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), 1\right\} \circ \min\left\{1, N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), 1, N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), 1\right\} \circ 1$$

$$N(T\alpha, T\alpha, \alpha, t) \geq \min\left\{1, N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), 1, \quad N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), 1\right\} \circ \min\left\{1, N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), 1, N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), 1\right\} \circ 1$$

$$N(T\alpha, T\alpha, \alpha, t) \geq \min\left\{N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), \quad N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right)\right\} \circ \min\left\{N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right)\right\} \circ 1$$

Hence,$N(T\alpha, T\alpha, \alpha, t) \geq N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right)$.

where $t > 0$ and Lemma 2.14 with $q \in \left(0, \frac{1}{3}\right)$.

It follows that $T = \alpha$.

To prove uniqueness, Let $T(\beta) = \beta$ for some $\beta \in X$, then

$$N(\beta, \beta, \alpha, t) = N(T\beta, T\beta, T\alpha, t)$$
$$\geq \min\left\{N\left(\beta, \beta, \alpha, \frac{t}{q}\right), N\left(T\beta, T\beta, \beta, \frac{t}{q}\right), N\left(T\alpha, T\alpha, \alpha, \frac{t}{q}\right), N\left(T\beta, T\beta, \alpha, \frac{t}{q}\right), N\left(\beta, \beta, T\alpha, \frac{3t}{q}\right)\right\}$$

[By (ii) of theorem (3.1)]

$$\geq \min\left\{N\left(\beta, \beta, \alpha, \frac{t}{q}\right), N\left(T\beta, T\beta, \beta, \frac{t}{q}\right), N\left(T\alpha, T\alpha, \alpha, \frac{t}{q}\right), N\left(T\beta, T\beta, \alpha, \frac{t}{q}\right), \left(N\left(\beta, \beta, \alpha, \frac{t}{q}\right) \circ N\left(\beta, \beta, \alpha, \frac{t}{q}\right) \circ N\left(T\alpha, T\alpha, \alpha, \frac{t}{q}\right)\right)\right\}$$

[By N_3 of definition 2.2]

Since, $\alpha \circ \beta \circ \gamma = \min\{\alpha,\beta,\gamma\}$

$$\geq \min\left\{N\left(\beta,\beta,\alpha,\frac{t}{q}\right), N\left(T\beta,T\beta,\beta,\frac{t}{q}\right), N\left(T\alpha,T\alpha,\alpha,\frac{t}{q}\right), N\left(T\beta,T\beta,\alpha,\frac{t}{q}\right), \min\left\{N\left(\beta,\beta,\alpha,\frac{t}{q}\right), N\left(\beta,\beta,\alpha,\frac{t}{q}\right), N\left(T\alpha,T\alpha,\alpha,\frac{t}{q}\right)\right\}\right\}$$

$$\geq \min\left\{N\left(\beta,\beta,\alpha,\frac{t}{q}\right), N\left(\beta,\beta,\beta,\frac{t}{q}\right), N\left(\alpha,\alpha,\alpha,\frac{t}{q}\right), N\left(\beta,\beta,\alpha,\frac{t}{q}\right), \min\left\{N\left(\beta,\beta,\alpha,\frac{t}{q}\right), N\left(\beta,\beta,\alpha,\frac{t}{q}\right), N\left(\alpha,\alpha,\alpha,\frac{t}{q}\right)\right\}\right\}$$

Hence,

$$N(\beta,\beta,\alpha,t) \geq \min\left\{\left\{N\left(\beta,\beta,\alpha,\frac{t}{q}\right), 1, 1, N\left(\beta,\beta,\alpha,\frac{t}{q}\right)\right\}, \min\left\{N\left(\beta,\beta,\alpha,\frac{t}{q}\right), N\left(\beta,\beta,\alpha,\frac{t}{q}\right), 1\right\}\right\}$$

$$N(\beta,\beta,\alpha,t) \geq \min\left\{N\left(\beta,\beta,\alpha,\frac{t}{q}\right), N\left(\beta,\beta,\alpha,\frac{t}{q}\right)\right\}$$

$$N(\beta,\beta,\alpha,t) \geq N\left(\beta,\beta,\alpha,\frac{t}{q}\right) \quad \text{[By lemma 2.14]}$$

:

:

$$\geq N\left(\beta,\beta,\alpha,\frac{t}{(q)^n}\right) \to 1 \text{ as } n \to \infty$$

Thus $\alpha = \beta$ and this complete the proof.

Corollary 3.2 Let $(X, N, \circ)$ be a complete N-FMS and let $: X \to X$. If for some $q \in \left(0,\frac{1}{3}\right)$,

(iii) $\lim_{t\to\infty} N(\alpha,\beta,\gamma,t) = 1$,

(iv) $N(T\alpha,T\alpha,T\beta,t) \geq N\left(\alpha,\alpha,\beta,\frac{t}{q}\right)$

Where $\circ$ is minimum t-norm.Then,T has a unique fixed point in X

Proof :-

If $\min\left\{N\left(\alpha,\alpha,\beta,\frac{t}{q}\right), N\left(T\alpha,T\alpha,\alpha,\frac{t}{q}\right), N\left(T\beta,T\beta,\beta,\frac{t}{q}\right), N\left(T\alpha,T\alpha,\beta,\frac{t}{q}\right), N\left(\alpha,\alpha,T\beta,\frac{3t}{q}\right)\right\} = N\left(\alpha,\alpha,\beta,\frac{t}{q}\right)$

Then form Theorem 3.1 proof is obvious

Remark 3.3 : Corollary 3.2 is the Banach contration principal in N Fuzzy metric space which is proved by Agrawal et al. [1] hence theorem 3.1 generalize Banach contration principal in N Fuzzy metric space

Example 3.4 Let X be a set of nonnegative real number and $(X, N, \circ)$ be the complete N-FMS where N is defined by

$N(\alpha,\beta,\gamma,t) = e^{\frac{-[|\alpha-\gamma|+|\beta-\gamma|]}{t}}\ \forall\, \alpha,\beta,\gamma \in X\ \&\ t > 0$

Let $T(\alpha) = \ln(1+\frac{\alpha}{4})$

Now, $N(T\alpha,T\alpha,T\beta,t) = e^{\frac{-(|T\alpha-T\beta|+|T\alpha-T\beta|)}{t}} = e^{\frac{-2|T\alpha-T\beta|}{t}}$

$$= e^{\frac{-2\ln\left(1+\frac{\alpha}{4}\right)-\ln\left(1+\frac{\beta}{4}\right)}{t}}$$

$\geq e^{\frac{-2\left(\left|\frac{\alpha}{4}-\frac{\beta}{4}\right|\right)}{t}}$ [Since $\ln(1+\alpha) \leq \alpha$ for each $\alpha \in \{0,\infty)$] ${}^{t}/_{4}$

$$= e^{\frac{-(2|\alpha-\beta|)}{t/4}} = N\left(\alpha,\alpha,\beta,\frac{t}{\frac{1}{4}}\right)$$

Hence, $N(T\alpha,T\alpha,T\beta,t) \geq N\left(\alpha,\alpha,\beta,\frac{t}{q}\right)$ where $q = \frac{1}{4}$

.

Hence, all the conditions of corollary (3.2) are satisfied and 0 is unique fixed point of T.

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References:

[1] Agrawal S, Malviya N.The Banach contration principle in N-Fuzzy Metric Space with application, accepted in conference , ICRAMDS-2024, MANIT, Bhopal, MP, India

[2] Banach, S.; Surles operations danseles ensembles abstracts at leur applications aux equations integrals Fund. math., 3(1922), 133-18

[3] George A., Veeramani P., On some results in fuzzy metric spaces, Fuzzy Set Syst., 64 (1994), 395–399. https://doi.org/10.1016/0165-0114(94)90162-7

[4] Malviya N. The N-Fuzzy Metric Spaces and Mappings with Application, Fasciculi Mathematici, https://doi.org/10.1515/fascmath-2015-0019

[5] Rakic, D., Mukheimer, A., Dosenovic, T., Mitrovic, D. Z., On Some new fixed point results in fuzzy b-metric spaces, Journal of inequalities and applications, (2020), 2020:99. https://doi.org/10.1186/s13660-020-02371-3.

[6] Schweizer, B., Sklar, A., Statistical metric spaces, Pacific J. Math., 10 (1960), 314–334.

[7] Sedghi S., Shobe N., Aliouche A., A generalization of fixed-point theorems in S-metric spaces, Mat. Vesn., 64 (2012), 258–266.

[8] Shikha Shende, G. Agrawal and N. Malviya, Some Fixed Point Theorem For Asymptotically Regular Maps in N-Fuzzy Metric Space. e-ISSN 2456-7701 Journal of Science and Technological Researches (JSTR) Vol. 3 Issue No.3, July-September 2021.