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On the Generalize Contraction in *N*-Fuzzy Metric Space with Applications

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Abstract: In this paper, the generalize q-contraction introdued in *N*-fuzzy metric space (*N*-FMS) and proved a fixed point theorem. Application of result also given in integral theory.

Keywords: *N*-Fuzzy metric space, generalize q-contraction, Fixed point, Integral equation.

AMS subject classifications: 47H10; 54H25

1.Introduction:

In the mathematical analysis many generalization of fuzzy metric spaces are exist.
In 2015, Malviya N. [4] introduced the notion of *N*-fuzzy metric space (*N*-FMS) (which is the generalization of fuzzy metric space [3] and S-metric space [7]) and pseudo *N*-fuzzy metric space with some topological and convergence properties. In 2021, various results related to asymptotically regular maps and sequences are proved by Shikha et al. [8]. Very recently Agrawal et al [1] investigated q- contraction in *N*-Fuzzy metric space and given an application of Banach contraction theorem in integral theory.

In this paper, we introduce generalize q-contraction in structure of N-Fuzzy metric space and proved a fixed point theorem which extend and generalize the theorem 2.5 of Rakic D. et.al. [5] and Agrawal et al.[1].We have also given the application of generalize q-contraction in integral theory.

2.Preliminaries:

We recall the following related definitions of *N*-fuzzy metric space (*N*-FMS).

Definition 2.1 [6]:A map $\circ: [0,1]^3 \rightarrow [0,1]$ is called continuous t-norm if it satisfies the following conditions:

(1) $\circ(\alpha,1,1) = \alpha, \circ(0,0,0) = 0$.
(2) $\circ(\alpha,\beta,\gamma) = \circ(\gamma,\alpha,\beta) = \circ(\beta,\gamma,\alpha)$.
(3) $\circ(\alpha_1,\beta_1,\gamma_1) \geq \circ(\alpha_2,\beta_2,\gamma_2)$ for $\alpha_1 \geq \alpha_2, \beta_1 \geq \beta_2, \gamma_1 \geq \gamma_2$.

Examples of t-norm are

(1) $\alpha \circ \beta \circ \gamma = \alpha \cdot \beta \cdot \gamma$(Product t-norm).
(2) $\alpha \circ \beta \circ \gamma = \min\{\alpha,\beta,\gamma\}$ (Minimum$t$-norm).

Definition 2.2 [4]:A triplet $(X,N,\circ)$is N-FMS if X is an arbitrary nonempty set, $\circ$ is continuous t-norm, and N is a fuzzy set on $X^3 \times (0,\infty)$ satisfying the following conditions for all $\alpha,\beta,\gamma,a \in X$ and $r,k,t > 0$:

(N_1) $N(\alpha,\beta,\gamma,t) > 0$,
(N_2) $N(\alpha,\beta,\gamma,t) = 1$ iff $\alpha = \beta = \gamma$,
(N_3) $N(\alpha,\beta,\gamma,(r+k+t)) \geq N(\alpha,\alpha,a,r) \circ N(\beta,\beta,a,k) \circ N(\gamma,\gamma,a,t)$,
(N_4) $N(\alpha,\beta,\gamma,\cdot)(0,\infty) \rightarrow (0,1]$ is a continuous function.

$N(\alpha,\beta,\gamma,t)$ is considered as the degree of nearness of α,β and γ with respect to t.

Proposition 2.3 [4]: Let $(X,N,\circ)$ be an N-FMSs. then we have
$N(\alpha,\alpha,\beta,t)=N(\beta,\beta,\alpha\,,t)$, for all $\alpha,\beta\in X$ and $t>0$.

Definition 2.4[4]: Let $(X,N,\circ)$ be an N-FMS. A sequence $\{\alpha_n\}$ in $(X,N,\circ)$converges to $\alpha\in X$ if $N(\alpha_n,\alpha_n,\alpha,t)\to 1$ or $N(\alpha,\alpha,\alpha_n,t)\to 1$ as $n\to\infty$ for each $t>0$.which is written as

$$\lim_{n\to\infty} N(\alpha_n,\alpha_n,\alpha,t)\to 1 \text{ for all } t>0$$

Or

$$\lim_{n\to\infty} N(\alpha,\alpha,\alpha_n,t)\to 1 \text{ for all } t>0$$

That is, for $r>0$ and $t>0$ there exists $n_0\in N$ such that for all $n\geq n_0$,
$N(\alpha_n,\alpha_n,\alpha,t)>1-r$ or $N(\alpha,\alpha,\alpha_n\,,t)>1-r$.

Lemma 2.5[4]: Let $(X,N,\circ)$ be an N-FMS, where $\circ$ is minimum t-norm. Let $\{\alpha_n\}$ be a sequence in X. if $\{\alpha_n\}$ converges to α and $\{\alpha_n\}$ also converges to β then $\alpha=\beta$. That is if the limit of $\{\alpha_n\}$ exists, it is unique.

Definition 2.6[4]: Let $(X,N,\circ)$ be an N-FMS and $\{\alpha_n\}$ be a sequence in X is called Cauchy sequence, if for each $r>0$ and $t>0$, there exists $n_0\in N$ such that

$$N(\alpha_n,\alpha_n,\alpha_m,t)>1-r,$$

Or

$$N(\alpha_m,\alpha_m,\alpha_n,t)>1-r \text{ ,For all } n,m\ \geq n_0.$$

Definition 2.7[4]: Let $(X,N,\circ)$ be an N-FMS. If every Cauchy sequence $\{\alpha_n\}$ is convergent to a point $\in X$, then X is called a complete N-FMS.

Definition 2.8[4]: Let $(X,N,\circ)$ be an N-FMS. A subset A of X is said to be F-bounded if there exist $t>0$ and $0<r<1$ such that

$$N(\alpha,\alpha,\beta,t)>1-r, \text{ for all } \alpha,\beta\in A.$$

Definition 2.9[4]: Let $(X,N,\circ)$ be an N-FMS. A self map$f\colon X\to X$ is a fuzzy q-contraction if for all $\alpha,\beta\in X$ and for some $q\in(0,1)$, we have

$$N(f(\alpha),f(\alpha),f(\beta),qt)\ \geq N(\alpha,\alpha,\beta,t).$$

Lemma 2.10[4]: Let $(X,N,\circ)$ be an N-FMS, where $\circ$ is product t norm and $\{\alpha_n\}$ be a sequence in X. if $\{\alpha_n\}$ converges to α, then $\{\alpha_n\}$ is a Cauchy sequence.

Definition 2.11[4]**:**Let $(X,N,\circ)$ and $(X',N',\circ')$ be N-FMS. Then a function $T\colon X\to X'$ is said to be continuous at a point α we have $\{T\alpha_n\}$ is convergent to $T(\alpha)$.

Proposition 2.12[4]: Let $(X,N,\circ)$ be N-FMS and T be a fuzzy q-contraction. If any fixed point α of satisfies

$$N(\alpha,\alpha,\alpha,t)>0,$$

Then

$$N(\alpha,\alpha,\alpha,t)=1$$

Lemma 2.13[4]: Let $(X, N, \circ)$ be a N-FMS. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in X and suppose $\alpha_n \to \alpha, \beta_n \to \beta$, as $n \to \infty$ and $N(\alpha, \alpha, \beta, t_n) \to N(\alpha, \alpha, \beta, t)$ as $n \to \infty$. Then $N(\alpha_n \alpha_n, \beta_n, t_n) \to N(\alpha, \alpha, \alpha, t)$ as $n \to \infty$.

Lemma 2.14[4]: Let $(X, N, \circ)$ be a N-FMS. If there exists $q \in (0,1)$ such that $N(\alpha, \alpha, \beta, t) \geq N\left(\alpha, \alpha, \beta, \frac{t}{q}\right)$ for all $\alpha, \beta \in X$, $t > 0$ and $\lim_{t\to\infty} N(\alpha, \beta, \gamma, t) = 1$.Then $\alpha = \beta$.

3. Main Result:

Theorem 3.1 : Let $(X, N, \circ)$ be a complete N-FMS. A self map $T : X \to X$. If for all $\alpha, \beta \in X$ and for some $q \in (0, \frac{1}{3})$ we have $t > 0$

(i) $\lim_{t\to\infty} N(\alpha, \beta, \gamma, t) = 1$

(ii) $N(T\alpha, T\alpha, T\beta, t) \geq \min\{N\left(\alpha, \alpha, \beta, \frac{t}{q}\right), N\left(T\alpha, T\alpha, \alpha, \frac{t}{q}\right), N\left(T\beta, T\beta, \beta, \frac{t}{q}\right)\}$

Where $\circ$ is minimum t-norm. Then T has unique fixed point.

Proof: Let $\alpha_o \in X$ for $n > 0$, $t > 0$, Such that $\alpha_{n+1} = T\alpha_n, n \in N$.
We have

$$N(\alpha_n, \alpha_n, \alpha_{n+1}, t) = N(T\alpha_{n-1}, T\alpha_{n-1}, T\alpha_n, t)$$

$$\geq \min\left\{N\left(\alpha_{n-1}, \alpha_{n-1}, \alpha_n, \frac{t}{q}\right), N\left(T\alpha_{n-1}, T\alpha_{n-1}, \alpha_n, \frac{t}{q}\right), N\left(T\alpha_n, T\alpha_n, \alpha_n, \frac{t}{q}\right)\right\}$$

$$= \min\left\{N\left(\alpha_{n-1}, \alpha_{n-1}, \alpha_n, \frac{t}{q}\right), N\left(\alpha_n, \alpha_n, \alpha_n, \frac{t}{q}\right), N\left(\alpha_{n+1}, \alpha_{n+1}, \alpha_n, \frac{t}{q}\right)\right\}$$

$$= \min\left\{N\left(\alpha_{n-1}, \alpha_{n-1}, \alpha_n, \frac{t}{q}\right), 1, N\left(\alpha_{n+1}, \alpha_{n+1}, \alpha_n, \frac{t}{q}\right)\right\}$$

$$\Rightarrow N(\alpha_n, \alpha_n, \alpha_{n+1}, t) \geq \min\left\{N\left(\alpha_{n-1}, \alpha_{n-1}, \alpha_n, \frac{t}{q}\right), N\left(\alpha_{n+1}, \alpha_{n+1}, \alpha_n, \frac{t}{q}\right)\right\}$$

Case-I

If $N(\alpha_n, \alpha_n, \alpha_{n+1}, t) \geq N\left(\alpha_{n+1}, \alpha_{n+1}, \alpha_n, \frac{t}{q}\right)$
When $n \in N$, $t > 0$, then by lemma 2.14 of it follows that $\alpha_n = \alpha_{n+1}$ for $n \in N$.

Case-II

$$\begin{aligned} N(\alpha_n, \alpha_n, \alpha_{n+1}, t) &\geq N\left(\alpha_{n-1}, \alpha_{n-1}, \alpha_n, \frac{t}{q}\right) \\ &\geq N\left(\alpha_{n-2}, \alpha_{n-2}, \alpha_{n-1}, \frac{t}{q^2}\right) \\ &\vdots \\ &\geq N\left(\alpha_0, \alpha_0, \alpha_1, \frac{t}{q^n}\right) \end{aligned}$$

Hence, $N(\alpha_n, \alpha_n, \alpha_{n+1}, t) \geq N\left(\alpha_0, \alpha_0, \alpha_1, \frac{t}{q^n}\right)$.

Now, for Cauchy sequence $m, n > 0$ and $n > m$

$N(\alpha_n, \alpha_n, \alpha_{n+m}, t) \geq N\left(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{t}{3}\right) \circ N\left(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{t}{3}\right) \circ$
$N\left(\alpha_{n+m}, \alpha_{n+m}, \alpha_{n+1}, \frac{t}{3}\right)$[by N_{b3} of definition 2.2]

$$= N\left(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{t}{3}\right) \circ N\left(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{t}{3}\right) \circ N\left(\alpha_{n+1}, \alpha_{n+1}, \alpha_{n+m}, \frac{t}{3}\right)$$
[by symmetric property]

$$\geq N\left(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{t}{3}\right) \circ N\left(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{t}{3}\right) \circ N\left(\alpha_{n+1}, \alpha_{n+1}, \alpha_{n+2}, \frac{t}{(3)^2}\right) \circ N\left(\alpha_{n+1}, \alpha_{n+1}, \alpha_{n+2}, \frac{t}{(3)^2}\right) \circ N\left(\alpha_{n+m}, \alpha_{n+m}, \alpha_{n+2}, \frac{t}{(3)^2}\right)$$

$$= N\left(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{t}{3}\right) \circ N\left(\alpha_n, \alpha_n, \alpha_{n+1}, \frac{t}{3}\right) \circ N\left(\alpha_{n+1}, \alpha_{n+1}, \alpha_{n+2}, \frac{t}{(3)^2}\right) \circ N\left(\alpha_{n+1}, \alpha_{n+1}, \alpha_{n+2}, \frac{t}{(3)^2}\right) \circ N\left(\alpha_{n+2}, \alpha_{n+2}, \alpha_{n+m}, \frac{t}{(3)^2}\right)$$
[By symmetric property]

$$\geq N\left(\alpha_0, \alpha_0, \alpha_1, \frac{t}{q^n(3)}\right) \circ N\left(\alpha_0, \alpha_0, \alpha_1, \frac{t}{q^n(3)}\right) \circ N\left(\alpha_0, \alpha_0, \alpha_1, \frac{t}{q^{n+1}(3)^2}\right) \circ N\left(\alpha_0, \alpha_0, \alpha_1, \frac{t}{q^{n+1}(3)^2}\right) \dots\dots$$

By the condition (i) of 3.1and $q < \frac{1}{3}$, we get,
$\lim_{n\to\infty} N(\alpha_n, \alpha_n, \alpha_{m+n}, t) = 1 \circ 1 \circ 1 \circ 1 \dots\dots\dots = 1$.
Hence,$\{\alpha_n\}$ in Cauchy sequence. Since, $(X, N_b, \circ)$ is complete symmetric N-FMS, there exists $\alpha \in X$ such that
$\lim_{n\to\infty} \alpha_n = \alpha$………(3.1.1)

Now, we show that α is a fixed point of T

$$N(T\alpha, T\alpha, \alpha, t) \geq N\left(T\alpha, T\alpha, T\alpha_n, \frac{t}{3}\right) \circ N\left(T\alpha, T\alpha, T\alpha_n, \frac{t}{3}\right) \circ N\left(\alpha, \alpha, T\alpha_n, \frac{t}{3}\right)$$

$$\geq \min\left\{N_b\left(\alpha, \alpha, \alpha_n, \frac{t}{3q}\right), N_b\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), N_b\left(T\alpha_n, T\alpha_n, \alpha_n, \frac{t}{3q}\right)\right\} \circ \min\left\{N_b\left(\alpha, \alpha, \alpha_n, \frac{t}{3q}\right), N_b\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), N_b\left(T\alpha_n, T\alpha_n, \alpha_n, \frac{t}{3q}\right)\right\} \circ N_b\left(\alpha, \alpha, \alpha_{n+1}, \frac{t}{3q}\right)$$

$$= \min\left\{N\left(\alpha, \alpha, \alpha_n, \frac{t}{3q}\right), N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), N\left(\alpha_{n+1}, \alpha_{n+1}, \alpha_n, \frac{t}{3q}\right)\right\} \circ \min\left\{N\left(\alpha, \alpha, \alpha_n, \frac{t}{3q}\right), N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), N\left(\alpha_{n+1}, \alpha_{n+1}, \alpha_n, \frac{t}{3q}\right)\right\} \circ N\left(\alpha, \alpha, \alpha_{n+1}, \frac{t}{3q}\right)$$

Taking $n \to \infty$ and $t > 0$
$\geq \min\left\{1, N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), 1\right\} \circ \min\left\{1, N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), 1\right\} \circ 1 =$
$\min\left\{N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right), 1\right\}$
$\Longrightarrow N(T\alpha, T\alpha, \alpha, t) \geq N\left(T\alpha, T\alpha, \alpha, \frac{t}{3q}\right).$
This show that $T\alpha = \alpha$, that is α in fixed point of T.

Uniqueness: Let $T\alpha = \alpha$ and $T\beta = \beta$ for $\beta \in X$, then
By condition (ii) of 3.1, we get,

$$N(\alpha, \alpha, \beta, t) = N(T\alpha, T\alpha, T\beta, t)$$
$$\geq \min\left\{N\left(\alpha, \alpha, \beta, \frac{t}{q}\right), N\left(\alpha, \alpha, T\alpha, \frac{t}{q}\right), N\left(\beta, \beta, T\beta, \frac{t}{q}\right)\right\}$$
$$= \min\left\{N\left(\alpha, \alpha, \beta, \frac{t}{q}\right), 1,1\right\}$$
$$= N\left(\alpha, \alpha, \beta, \frac{t}{q}\right)$$
$$= N\left(T\alpha, T\alpha, T\beta, \frac{t}{q}\right)$$
$$\geq \min\left\{N\left(\alpha, \alpha, \beta, \frac{t}{q^2}\right), N\left(\alpha, \alpha, T\alpha, \frac{t}{q^2}\right), N\left(\beta, \beta, T\beta, \frac{t}{q^2}\right)\right\}$$
$$= \min\left\{N\left(\alpha, \alpha, \beta, \frac{t}{q^2}\right), 1,1\right\}$$
$$\geq N\left(\alpha, \alpha, \beta, \frac{t}{q^2}\right)$$
:
$$\Rightarrow N(\alpha, \alpha, \beta, t) \geq N\left(\alpha, \alpha, \beta, \frac{t}{q^n}\right) \to 1 \text{ as } n \to \infty$$

Thus $\alpha = \beta$, and this complete the proof.

Corollary 3.2: Let $(X, N, \circ)$ be a complete N-FMS with
$\lim_{t\to\infty} N(\alpha, \beta, \gamma, t) = 1$,
Where $\circ$ is minimum t-norm. And T be a fuzzy q-contraction. Then T has a unique fixed point.

Proof: In theorem 3.1

$$N(T\alpha, T\alpha, T\beta, t) \geq \min\left\{N\left(\alpha, \alpha, \beta, \frac{t}{q}\right), N\left(T\alpha, T\alpha, \alpha, \frac{t}{q}\right), N\left(T\beta, T\beta, \beta, \frac{t}{q}\right)\right\}$$

If we take the condition
$N\left(\alpha, \alpha, \beta, \frac{t}{q}\right) \leq N\left(T\alpha, T\alpha, \alpha, \frac{t}{q}\right)$ and $N\left(\alpha, \alpha, \beta, \frac{t}{q}\right) \leq N\left(T\beta, T\beta, \beta, \frac{t}{q}\right)$
Then,
$N(T\alpha, T\alpha, T\beta, t) \geq N\left(\alpha, \alpha, \beta, \frac{t}{q}\right)$
Which is the definition of q-contraction and if $q \in \left(0, \frac{1}{3}\right)$ then $q \in (0,1)$.
Hence, our theorem generalize theorem 3.1 of Agrawal et. al. [1] for minimum t norm.
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4.Application in Integral theory:

The first integral type of BCP was proved by Branciari[2]. Let $\theta: (0,\infty) \to (0,\infty)$ as
$\theta(t) = \int_0^t \phi(t)dt,\ \forall t > 0,$
be a nondecreasing and continuous function. Moreover, for each $r > 0, \phi(r) > 0$. It also implies that $\phi(t) = 0$ iff $t = 0$.

In following, we prove integral analogue of theorem 3.1in $N - FMS$.

Theorem 4.1: Let (X, N, o) be a complete $N - FMS$ and $T: X \to X$ be a map satisfying

$$\int_0^{N(T(\alpha),T(\alpha),T(\beta),qt)} \phi(t)dt,$$

$$\geq \int_0^{\min\{N\left(\alpha,\alpha,\beta,\frac{t}{q}\right),N\left(T\alpha,T\alpha,\alpha,\frac{t}{q}\right),N\left(T\,\beta,T\,\beta,\beta,\frac{t}{q}\right)\}} \phi(t)dt$$

For all $\alpha, \beta, \in X$, $\phi \in \theta$, and $q \in (0,1)$. Then T has a unique fixed point.

Proof: By taking $\phi(1) = 1$ and applying theorem 3.1, we obtain the result.

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