

The Intuitionistic Menger b-Metric Space

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Abstract: In this manuscript, we introduce the notion of intuitionistic Menger b-metric space, (IMb-MS) which is the hybrid structure of Menger space and b-metric space. We also prove an example of IMb-MS. In the last section we prove a Banach contraction theorem in the setting of intuitionistic Menger b-metric space.

Keywords and Phrases: Menger b-metric space, Intuitionistic Menger b-metric space, Fixed point.

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Introduction:

The theory of probabilistic metric spaces was initiated by Karl Menger [9] in 1942, where the distance between two points is described by a distribution function instead of a nonnegative real number. This pioneering idea laid the foundation for the study of uncertainty in metric structures. Subsequently, Schweizer and Sklar [10,11] significantly advanced the theory by introducing Menger spaces, which have become an important framework in probabilistic analysis and fixed-point theory.

In 1989, Bakhtin [2] introduced the notion of b-metric spaces as a generalization of metric spaces. Later, Czerwik [6] studied contraction mappings in b-metric spaces and established several fixed-point results. Due to their flexible structure, b-metric spaces have attracted considerable attention in nonlinear analysis and its applications.

The concept of intuitionistic Menger spaces was introduced by Kutukcu et al. [7] by combining intuitionistic fuzzy sets with Menger spaces. Later, Mbarki and Oubrahim [8] introduced b-Menger spaces, extending the classical Menger space framework to b-metric spaces. Several important contributions to fixed-point theory in Menger space have also been made by Choudhary et al. [3–5].

Motivated by these developments, in this paper we introduce the concept of intuitionistic Menger b-metric space with an example. Also, we proved the Banach contraction theorem in this setting.

2. Preliminaries

For the reader convenience some definitions and results are recalled.

Definition 2.1.[10] A t -norm is a function $*$: $[0,1]^2 \rightarrow [0,1]$ which satisfies the following conditions

- (i) $T(a, 1) = a$, $T(0, 0) = 0$;
- (ii) $T(a, b) = T(b, a)$;
- (iii) $T(T(a, b), c) = T(a, T(b, c))$;

(iv) $T(c, d) \geq T(a, b)$ whenever $c \geq a$ and $d \geq b$.

The following defines three basic t-norms:

1. The minimum t -norm, $a * b = \min\{a, b\}$
2. The Product t -norm, $a * b = ab$
3. The Lukasiewicz t -norm, $a * b = \max\{a + b - 1, 0\}$

Definition 2.2. [7] A binary operation $\phi: [0, 1]^2 \rightarrow [0, 1]$ is called a continuous triangular co-norm (t -conorm) if it satisfies the following conditions:

- (1) ϕ is associative and commutative;
- (2) ϕ is continuous;
- (3) $a \phi 0 = a, \forall a \in [0, 1]$;
- (4) $a \phi b \leq c \phi d$, whenever $a \leq c$ and $b \leq d \forall a, b, c, d \in [0, 1]$.

Three basic t -conorms are given below:

- (1) $a \diamond_1 b = \min(a + b, 1)$,
- (2) $a \diamond_2 b = (a + b - ab)$,
- (3) $a \diamond_3 b = \max(a, b)$

Definition 2.3. [8] Let T be a continuous t -norm, X be a nonempty set, and $k \geq 1$ be a fixed real number. A mapping from $X \times X$ into the set of distribution functions is called a Menger b -metric on X if and only if the following conditions hold for every $\alpha, \beta, \gamma \in X$:

- (i) $\mathcal{M}_{b(\alpha, \beta)}(0) = 0$
- (ii) $\mathcal{M}_{b(\alpha, \beta)}(t) = 1, \forall t > 0$ if and only if $\alpha = \beta$
- (iii) $\mathcal{M}_{b(\alpha, \beta)}(t) = \mathcal{M}_{b(\beta, \alpha)}(t), \forall t > 0$
- (iv) $\mathcal{M}_{b(\alpha, \gamma)}(k(t + u)) \geq T\left[\mathcal{M}_{b(\alpha, \beta)}(t), \mathcal{M}_{b(\beta, \gamma)}(u)\right], \forall t, s \geq 0$.

The quadruple $(X, \mathcal{M}_b, o, k)$ is said to be a Menger b -metric space.

Definition 2.4. [10] A distance distribution function is a function, $\mathcal{M}: \mathbb{R} \rightarrow \mathbb{R}^+$ which is left continuous on $\mathbb{R}$, non-decreasing and $\inf_{t \in \mathbb{R}} \mathcal{M}(t) = 0, \sup_{t \in \mathbb{R}} \mathcal{M}(t) = 1$. we shall be denoted by D the family of all distance distribution functions and by $\mathcal{H}$ a special distance distribution function in D given by

$$\mathcal{H}(t) = \begin{cases} 1 & , \text{if } t > 0 \\ 0 & , \text{if } t = 0 \end{cases}$$

Definition 2.5. [10] A non-distance distribution function is function $\mathcal{L}: \mathbb{R} \rightarrow \mathbb{R}^+$ which is left continuous on $\mathbb{R}$, non-increasing and $\inf_{t \in \mathbb{R}} \mathcal{L}(t) = 1, \sup_{t \in \mathbb{R}} \mathcal{L}(t) = 0$. We shall denote by E the family of all non-distance distribution functions and by G a special non distance distribution function in E given by

$$G(t) = \begin{cases} 1 & , \text{if } t < 0 \\ 0 & , \text{if } t > 0 \end{cases}$$

Main Results:

In this section, we introduce the concept of intuitionistic Menger b -metric space with example.

Definition 3.1. The 6-tuple $(X, \mathcal{M}, \mathcal{L}, *, \diamond, b)$ is said to be an intuitionistic Menger b -metric space, if X is an arbitrary set, $b \geq 1$ is given real number, $*$ is a continuous t -norm, $\diamond$ is continuous t -co-norm, $\mathcal{M}$ is a probabilistic distance and $\mathcal{L}$ is a probabilistic non-distance on X satisfying the following conditions: for $\forall \alpha, \beta, \gamma \in X$ and $t, s \geq 0$

1. $\mathcal{M}_{(\alpha, \beta)}(t) + \mathcal{L}_{(\alpha, \beta)}(t) \leq 1$;
2. $\mathcal{M}_{(\alpha, \beta)}(0) = 0$;
3. $\mathcal{M}_{(\alpha, \beta)}(t) = \mathcal{H}(t)$, if and only if $\alpha = \beta$;
4. $\mathcal{M}_{(\alpha, \beta)}(t) = \mathcal{M}_{(\beta, \alpha)}(t)$;
5. $\mathcal{M}_{(\alpha, \gamma)}(t+u) \geq \mathcal{M}_{(\alpha, \beta)}\left(\frac{t}{b}\right) * \mathcal{M}_{(\beta, \gamma)}\left(\frac{u}{b}\right)$;
6. $\mathcal{L}_{(\alpha, \beta)}(0) = 1$;
7. $\mathcal{L}_{(\alpha, \beta)}(t) = G(t)$ if and only if $\alpha = \beta$;
8. $\mathcal{L}_{(\alpha, \beta)}(t) = \mathcal{L}_{(\beta, \alpha)}(t)$;
9. $\mathcal{L}_{(\alpha, \gamma)}(t+u) \leq \mathcal{L}_{(\alpha, \beta)}\left(\frac{t}{b}\right) \diamond \mathcal{L}_{(\beta, \gamma)}\left(\frac{u}{b}\right)$;

Definition 3.2. Let $(X, \mathcal{M}, \mathcal{L}, *, \diamond, b)$ be an intuitionistic Menger b -metric space.

- (a) A sequence $\{\alpha_n\}$ in X is said to be convergent if there exists $\alpha \in X$ such that $\lim_{n \rightarrow \infty} \mathcal{M}_{(\alpha_n, \alpha)}(t) = 1$ and $\lim_{n \rightarrow \infty} \mathcal{L}_{(\alpha_n, \alpha)}(t) = 0$, $\forall t > 0$. In this case α is called the limit of the sequence $\{\alpha_n\}$ and we write $\lim_{n \rightarrow \infty} \alpha_n = \alpha$, or $\alpha_n \rightarrow \alpha$.
- (b) A sequence $\{\alpha_n\}$ in $(X, \mathcal{M}, \mathcal{L}, *, \diamond, b)$ is said to be a Cauchy sequence if for every $\alpha_n \in X$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}_{(\alpha_n, \alpha_m)}(t) > 1 - \epsilon$ and $\mathcal{L}_{(\alpha_n, \alpha_m)}(t) < \epsilon$, $\forall m, n \geq n_0$ and $t > 0$.
- (c) The space X is said to be complete if every Cauchy sequence converges in it.

Example 3.3. Let $X = (0, \infty)$. Define a CTN $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ by $g * h = gh$ and a CTCN $\diamond$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ by $g \diamond h = \max\{g, h\}$ and also define $\mathcal{M}_b$ and $\mathcal{L}_b$ by

$$\mathcal{M}_b(\alpha, \beta)(t) = \begin{cases} [e^{\frac{(\alpha-\beta)^2}{t}}]^{-1}, & t > 0 \\ 0, & t = 0 \end{cases}$$

$$\mathcal{L}_b(\alpha, \beta)(t) = \begin{cases} 1 - [e^{\frac{(\alpha-\beta)^2}{t}}]^{-1}, & t > 0 \\ 1, & t = 0 \end{cases}$$

for all $\alpha, \beta \in X, t > 0$. Then it is an intuitionistic Menger b -metric space.

Proof. Now we prove (5) and (9)

$$(\alpha - \gamma)^2 \leq \left(\frac{b(t+s)}{t}\right) (\alpha - \beta)^2 + \left(\frac{b(t+s)}{s}\right) (\beta - \gamma)^2,$$

where b is an arbitrary integer, we have

$$\frac{(\alpha - \gamma)^2}{b(t+s)} \leq \frac{(\alpha - \beta)^2}{t} + \frac{(\beta - \gamma)^2}{s}.$$

That is,

$$e^{\frac{(\alpha - \gamma)^2}{b(t+s)}} \leq e^{\frac{(\alpha - \beta)^2}{t}} \cdot e^{\frac{(\beta - \gamma)^2}{s}}.$$

Since e^e is an increasing function, for $e > 0$, we have

$$[e^{\frac{(\alpha - \gamma)^2}{b(t+s)}}]^{-1} \geq [e^{\frac{(\alpha - \beta)^2}{t}}]^{-1} \cdot [e^{\frac{(\beta - \gamma)^2}{s}}]^{-1},$$

$$\mathcal{M}_{b(\alpha, \gamma)}(b(t+s)) \geq \mathcal{M}_{b(\alpha, \beta)}(t) * \mathcal{M}_{b(\beta, \gamma)}(s)$$

for all $\alpha, \beta, \gamma \in X, t, s > 0$. Hence (5) hold

Since $(\alpha - \gamma)^2 \geq \max\{(\alpha - \beta)^2, (\beta - \gamma)^2\}$,

$$\frac{(\alpha - \gamma)^2}{b(t+s)} \leq \max \left\{ \frac{(\alpha - \beta)^2}{t}, \frac{(\beta - \gamma)^2}{s} \right\},$$

where b is an arbitrary integer. Since e^e is an increasing function, for $e > 0$, we have

$$e^{\frac{(\alpha-\gamma)^2}{b(t+s)}} \leq \max \left\{ e^{\frac{(\alpha-\beta)^2}{t}}, e^{\frac{(\beta-\gamma)^2}{s}} \right\}.$$

So
$$\left[e^{\frac{(\alpha-\gamma)^2}{b(t+s)}} \right]^{-1} \geq \max \left\{ \left[e^{\frac{(\alpha-\beta)^2}{t}} \right]^{-1}, \left[e^{\frac{(\beta-\gamma)^2}{s}} \right]^{-1} \right\}$$

That is
$$1 - \left[e^{\frac{(\alpha-\gamma)^2}{b(t+s)}} \right]^{-1} \leq \max \left\{ 1 - \left[e^{\frac{(\alpha-\beta)^2}{t}} \right]^{-1}, 1 - \left[e^{\frac{(\beta-\gamma)^2}{s}} \right]^{-1} \right\}$$

$$\mathcal{L}_{b(\alpha,\gamma)}b(t+s) \leq \mathcal{L}_{b(\alpha,\beta)}(t) * \mathcal{L}_{b(\beta,\gamma)}(s)$$

for all $\alpha, \beta, \gamma \in X, t, s > 0$. Hence (9) hold.

The Banach Contraction Theorem:

In this section, we prove a unique common fixed point theorem which is the extension of Banach contraction theorem in the structure of intuitionistic Menger b -metric space.

Theorem 3.4. Let $(X, \mathcal{M}, \mathcal{L}, *, \diamond, b)$ be a complete intuitionistic Menger b -metric space such that $\lim_{n \rightarrow \infty} \mathcal{M}_{b(\alpha,\beta)}(t) = 1, \lim_{n \rightarrow \infty} \mathcal{L}_{b(\alpha,\beta)}(t) = 0$ for all $\alpha, \beta \in X, t > 0$ and $T: X \rightarrow X$ be a mapping satisfying the conditions

$$\mathcal{M}_{b(T\alpha,T\beta)}(at) \geq \mathcal{M}_{b(\alpha,\beta)}(t) \text{ and } \mathcal{L}_{b(T\alpha,T\beta)}(at) \leq \mathcal{L}_{b(\alpha,\beta)}(t) \dots\dots\dots (3.4.1)$$

for all $\alpha, \beta \in X, t > 0$, where $a \in (0,1)$. Then T has a unique fixed point $w \in X$ and $\mathcal{M}_{b(w,w)}(t) = 1, \mathcal{L}_{b(w,w)}(t) = 0$, for all $t > 0$.

Proof. Let $(X, \mathcal{M}, \mathcal{L}, *, \diamond, b)$ be a complete intuitionistic Menger b -metric space. For a given element $\alpha_0 \in X$, define a sequence $\{\alpha_n\}$ in X by

$$\alpha_1 = T\alpha_0, \alpha_2 = T^2\alpha_0 = T\alpha_1 \dots\dots\dots, \alpha_n = T^n\alpha_0 = T\alpha_{n-1} \text{ for all } n \in \mathbb{N}$$

If $\alpha_n = \alpha_{n-1}$ for some $n \in \mathbb{N}$, then α_n is a fixed point of T . We assume that $\alpha_n \neq \alpha_{n-1}$ for all $n \in \mathbb{N}$. For all $t > 0$ and $n \in \mathbb{N}$, we get from (3.4.1) that

$$\mathcal{M}_{b(\alpha_n,\alpha_{n+1})}(t) \geq \mathcal{M}_{b(\alpha_{n+1},\alpha_n)}(at) = \mathcal{M}_{b(T\alpha_n,T\alpha_{n-1})}(at) \geq \mathcal{M}_{b(\alpha_n,\alpha_{n-1})}(at)$$

and

$$\mathcal{L}_{b(\alpha_n,\alpha_{n+1})}(t) \leq \mathcal{L}_{b(\alpha_{n+1},\alpha_n)}(at) = \mathcal{L}_{b(T\alpha_n,T\alpha_{n-1})}(at) \leq \mathcal{L}_{b(\alpha_n,\alpha_{n-1})}(at)$$

for all $n \in \mathbb{N}$ and $t > 0$. Therefore, by applying the above expression, we can deduce that

$$\begin{aligned} \mathcal{M}_{b(\alpha_{n+1},\alpha_n)}(t) &\geq \mathcal{M}_{b(\alpha_{n+1},\alpha_n)}(at) = \mathcal{M}_{b(T\alpha_n,T\alpha_{n-1})}(at) \geq \mathcal{M}_{b(\alpha_n,\alpha_{n-1})}(t) \dots\dots\dots (3.4.2) \\ &= \mathcal{M}_{b(T\alpha_{n-1},T\alpha_{n-2})}(t) \geq \mathcal{M}_{b(\alpha_{n-1},\alpha_{n-2})}\left(\frac{t}{a}\right) \geq \dots \geq \mathcal{M}_{b(\alpha_1,\alpha_0)}\left(\frac{t}{a^n}\right) \end{aligned}$$

and

$$\mathcal{L}_{b(\alpha_{n+1},\alpha_n)}(t) \leq \mathcal{L}_{b(\alpha_{n+1},\alpha_n)}(at) = \mathcal{L}_{b(T\alpha_n,T\alpha_{n-1})}(at) \leq \mathcal{L}_{b(\alpha_n,\alpha_{n-1})}(t) \dots\dots\dots (3.4.3)$$

$$= \mathcal{L}_{b(T\alpha_{n-1},T\alpha_{n-2})}(t) \leq \mathcal{L}_{b(\alpha_{n-1},\alpha_{n-2})}\left(\frac{t}{a}\right) \leq \dots \leq \mathcal{L}_{b(\alpha_1,\alpha_0)}\left(\frac{t}{a^n}\right)$$

for all $n \in \mathbb{N}, p \geq 1$ and $t > 0$. Thus, we have

$$\mathcal{M}_{b(\alpha_n, \alpha_{n+1})}(t) \geq \mathcal{M}_{b(\alpha_n, \alpha_{n+1})}\left(\frac{t}{b}\right) * \mathcal{M}_{b(\alpha_{n+1}, \alpha_{n+p})}\left(\frac{t}{b}\right)$$

and

$$\mathcal{L}_{b(\alpha_n, \alpha_{n+1})}(t) \leq \mathcal{L}_{b(\alpha_n, \alpha_{n+1})}\left(\frac{t}{b}\right) \diamond \mathcal{L}_{b(\alpha_{n+1}, \alpha_{n+p})}\left(\frac{t}{b}\right).$$

Continuing in this way, we get

$$\mathcal{M}_{b(\alpha_n, \alpha_{n+p})}(t) \geq \mathcal{M}_{b(\alpha_n, \alpha_{n+1})}\left(\frac{t}{b}\right) * \mathcal{M}_{b(\alpha_{n+1}, \alpha_{n+2})}\left(\frac{t}{b^2}\right) * \dots * \mathcal{M}_{b(\alpha_{n+p-1}, \alpha_{n+p})}\left(\frac{t}{b^{p-1}}\right)$$

and

$$\mathcal{L}_{b(\alpha_n, \alpha_{n+p})}(t) \leq \mathcal{L}_{b(\alpha_n, \alpha_{n+1})}\left(\frac{t}{b}\right) \diamond \mathcal{L}_{b(\alpha_{n+1}, \alpha_{n+2})}\left(\frac{t}{b^2}\right) \diamond \dots \diamond \mathcal{L}_{b(\alpha_{n+p-1}, \alpha_{n+p})}\left(\frac{t}{b^{p-1}}\right)$$

Using (3.4.2) and (3.4.3) in the above inequality, we have

$$\mathcal{M}_{b(\alpha_n, \alpha_{n+p})}(t) \geq \mathcal{M}_{b(\alpha_0, \alpha_1)}\left(\frac{t}{ba^n}\right) * \mathcal{M}_{b(\alpha_0, \alpha_1)}\left(\frac{t}{b^2a^{n+1}}\right) * \dots * \mathcal{M}_{b(\alpha_0, \alpha_1)}\left(\frac{t}{b^{p-1}a^{n+p-1}}\right) \dots \quad (3.4.4)$$

and

$$\mathcal{L}_{b(\alpha_n, \alpha_{n+p})}(t) \leq \mathcal{L}_{b(\alpha_0, \alpha_1)}\left(\frac{t}{ba^n}\right) \diamond \mathcal{L}_{b(\alpha_0, \alpha_1)}\left(\frac{t}{b^2a^{n+1}}\right) \diamond \dots \diamond \mathcal{L}_{b(\alpha_0, \alpha_1)}\left(\frac{t}{b^{p-1}a^{n+p-1}}\right) \dots \quad (3.4.5)$$

Here b is an arbitrary positive integer.

We know that $\lim_{n \rightarrow \infty} \mathcal{M}_{b(\alpha, \beta)}(t) = 1, \lim_{n \rightarrow \infty} \mathcal{L}_{b(\alpha, \beta)}(t) = 0$ for all $\alpha, \beta \in X, t > 0$, where $a \in (0, 1)$. It follows from (3.4.4) and (3.4.5) that

$$\lim_{n \rightarrow \infty} \mathcal{M}_{b(\alpha_n, \alpha_{n+p})}(t) = 1 * 1 * \dots * 1 = 1 \text{ for all } t > 0, p \geq 1$$

and

$$\lim_{n \rightarrow \infty} \mathcal{L}_{b(\alpha_n, \alpha_{n+p})}(t) = 0 \diamond 0 \diamond \dots \diamond 0 = 0 \text{ for all } t > 0, p \geq 1.$$

Hence $\{\alpha_n\}$ is a Cauchy sequence.

The completeness of the intuitionistic Menger b -metric space $(X, \mathcal{M}, \mathcal{L}, *, \diamond, b)$ ensures that there exists $w \in X$ such that

$$\lim_{n \rightarrow \infty} \mathcal{M}_{b(\alpha_n, w)}(t) = \lim_{n \rightarrow \infty} \mathcal{M}_{b(\alpha_n, \alpha_{n+p})}(t) = \mathcal{M}_{b(w, w)}(t) = 1, \text{ for all } t > 0, p \geq 1 \dots \quad (3.4.6)$$

and

$$\lim_{n \rightarrow \infty} \mathcal{L}_{b(\alpha_n, w)}(t) = \lim_{n \rightarrow \infty} \mathcal{L}_{b(\alpha_n, \alpha_{n+p})}(t) = \mathcal{L}_{b(w, w)}(t) = 0. \text{ for all } t > 0, p \geq 1 \dots \quad (3.4.7)$$

Now, we show that $w \in X$ is a fixed point of T . We have

$$\begin{aligned} \mathcal{M}_{b(w, Tw)}(t) &\geq \mathcal{M}_{b(w, \alpha_{n+p})}\left(\frac{t}{2b}\right) * \mathcal{M}_{b(\alpha_{n+1}, Tw)}\left(\frac{t}{2b}\right) \\ &= \mathcal{M}_{b(w, \alpha_{n+1})}\left(\frac{t}{2b}\right) * \mathcal{M}_{b(T\alpha_n, Tw)}\left(\frac{t}{2b}\right) \\ &\geq \mathcal{M}_{b(w, \alpha_{n+1})}\left(\frac{t}{2b}\right) * \mathcal{M}_{b(\alpha_n, w)}\left(\frac{t}{2ba}\right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_{b(w, Tw)}(t) &\leq \mathcal{L}_{b(w, \alpha_{n+p})}\left(\frac{t}{2b}\right) \diamond \mathcal{L}_{b(\alpha_{n+1}, Tw)}\left(\frac{t}{2b}\right) \\ &= \mathcal{L}_{b(w, \alpha_{n+1})}\left(\frac{t}{2b}\right) \diamond \mathcal{L}_{b(T\alpha_n, Tw)}\left(\frac{t}{2b}\right) \\ &\leq \mathcal{L}_{b(w, \alpha_{n+1})}\left(\frac{t}{2b}\right) \diamond \mathcal{L}_{b(\alpha_n, w)}\left(\frac{t}{2ba}\right) \end{aligned}$$

For all $t > 0$. Taking the limit as $n \rightarrow \infty$, and by (3.4.6) and (3.4.7), we get

$$\mathcal{M}_{b(w, Tw)}(t) = 1 * 1 = 1$$

and

$$\mathcal{L}_{b(w,Tw)}(t) = 0 \diamond 0 = 0.$$

Therefore, w is a fixed point of T and $\mathcal{M}_{b(w,w)}(t) = 1, \mathcal{L}_{b(w,w)}(t) = 0$, for all $t > 0$. Now, we investigate the uniqueness of fixed point. For this, assume that v and w are two fixed point of T . Then by (3.4.1), we have

$$\mathcal{M}_{b(w,v)}(t) = \mathcal{M}_{b(Tw,Tv)}(t) \geq \mathcal{M}_{b(w,v)}\left(\frac{t}{a}\right)$$

and

$$\mathcal{L}_{b(w,v)}(t) = \mathcal{L}_{b(Tw,Tv)}(t) \leq \mathcal{L}_{b(w,v)}\left(\frac{t}{a}\right)$$

for all $t > 0$. Thus, we obtain

$$\mathcal{M}_{b(w,v)}(t) \geq \mathcal{M}_{b(w,v)}\left(\frac{t}{a^n}\right) \text{ for all } n \in \mathbb{N}$$

and

$$\mathcal{L}_{b(w,v)}(t) \leq \mathcal{L}_{b(w,v)}\left(\frac{t}{a^n}\right) \text{ for all } n \in \mathbb{N}$$

Taking the limit as $n \rightarrow \infty$ and using $\lim_{n \rightarrow \infty} \mathcal{M}_{b(\alpha,\beta)}(t) = 1$, $\lim_{n \rightarrow \infty} \mathcal{L}_{b(\alpha,\beta)}(t) = 0$, we get $w = v$. Hence the fixed point is unique.

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