The Intuitionistic Menger b-Metric Space

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Abstract: In this manuscript, we introduce the notion of intuitionistic Menger b-metric space, (IMb-MS) which is the hybrid structure of Menger space and b-metric space. We also prove an example of IMb-MS. In the last section we prove a Banach contraction theorem in the setting of intuitionistic Menger *b*-metric space.

Keywords and Phrases: Menger *b*-metric space, Intuitionistic Menger *b*-metric space, Fixed point.

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Introduction:

The theory of probabilistic metric spaces was initiated by Karl Menger [9] in 1942, where the distance between two points is described by a distribution function instead of a nonnegative real number. This pioneering idea laid the foundation for the study of uncertainty in metric structures. Subsequently, Schweizer and Sklar [10,11] significantly advanced the theory by introducing Menger spaces, which have become an important framework in probabilistic analysis and fixed-point theory.

In 1989, Bakhtin [2] introduced the notion of b-metric spaces as a generalization of metric spaces. Later, Czerwik [6] studied contraction mappings in b-metric spaces and established several fixed-point results. Due to their flexible structure, b-metric spaces have attracted considerable attention in nonlinear analysis and its applications.

The concept of intuitionistic Menger spaces was introduced by Kutukcu et al. [7] by combining intuitionistic fuzzy sets with Menger spaces. Later, Mbarki and Oubrahim [8] introduced b-Menger spaces, extending the classical Menger space framework to b-metric spaces. Several important contributions to fixed-point theory in Menger space have also been made by Choudhary et al. [3–5].

Motivated by these developments, in this paper we introduce the concept of intuitionistic Menger b-metric space with an example. Also, we proved the Banach contraction theorem in this setting.

2. Preliminaries

For the reader convenience some definitions and results are recalled.

Definition 2.1.[10] A *t*-norm is a function $*: [0,1]^2 \to [0,1]$ which satisfies the following conditions

(i) $T(a,1) = a$, $T(0,0) = 0$;

(ii) $T(a,b) = T(b,a)$;

(iii) $T(T(a,b),c) = T(a,T(b,c))$;

(iv) $T(c,d) \geq T(a,b)$ *whenever* $c \geq a$ *and* $d \geq b$.

The following defines three basic t-norms:

1. The minimum *t*-norm, $a * b = \min\{a,b\}$
2. The Product *t*-norm, , $a * b = ab$
3. The Lukasiewicz *t*-norm , $a * b = \max\{a + b - 1, 0\}$

Definition 2.2. [7] A binary operation $\lozenge$: $[0,1]^2 \rightarrow [0,1]$ is called a continuous triangular co-norm (*t*-conorm) if it satisfies the following conditions:

(1) $\lozenge$ is associative and commutative;

(2) $\lozenge$ is continuous;

(3) $a \lozenge 0 = a$, $\forall$, $a \in [0, 1]$;

(4) $a \lozenge b \leq c \lozenge d$, whenever $a \leq c$ and $b \leq d$ $\forall$ $a, b, c, d \in [0, 1]$.

Three basic *t*-conorms are given below:

(1) $a \lozenge_1 b$ = min $(a + b, 1)$,

(2) $a \lozenge_2 b = (a + b - ab)$,

(3) $a \lozenge_3 b$ = max (a, b)

Definition 2.3. [8] Let Tbe a continuous t-norm, Xbe a nonempty set, and $k \geq 1$ be a fixed real number. A mapping from $X \times X$ into the set of distribution functions is called a Menger b-metric on Xif and only if the following conditions hold for every $\alpha, \beta, \gamma \in X$:
(i) $\mathcal{M}_{b(\alpha,\beta)}(0) = 0$

(ii) $\mathcal{M}_{b(\alpha,\beta)}(t) = 1$, $\forall\, t > 0$ if and only if $\alpha = \beta$

(iii) $\mathcal{M}_{b(\alpha,\beta)}(t) = \mathcal{M}_{b(\beta,\alpha)}(t), \forall\, t > 0$

(iv) $\mathcal{M}_{b(\alpha,\gamma)}(k(t+u)) \geq T\left[\left(\mathcal{M}_{b(\alpha,\beta)}(t), \mathcal{M}_{b(\beta,\gamma)}(u)\right)\right], \forall\, t,s \geq 0.$

The quadruple $(X, \mathcal{M}_b, o, k)$ is said to be a Menger *b*-metric space.

Definition 2.4.[10] A distance distribution function is a function, $\mathcal{M}$: $\mathbb{R} \rightarrow \mathbb{R}+$ which is left continuous on $\mathbb{R}$, non-decreasing and $inf_{t\in\mathbb{R}}\mathcal{M}(t) = 0$, $\sup_{t\in\mathbb{R}}\mathcal{M}(t) = 1$. we shall be denoted by D the family of all distance distribution functions and by $\mathcal{H}$ a special distance distribution function in D given by

$$\mathcal{H}(t) = \begin{cases} 1 & ,\text{if } t > 0 \\ 0 & ,\text{if } t = 0 \end{cases}$$

Definition 2.5.[10] A non-distance distribution function is function $\mathcal{L}$: $\mathbb{R} \rightarrow \mathbb{R}+$ which is left continuous on $\mathbb{R}$, non-increasing and $inf_{t\in\mathbb{R}}\mathcal{L}(t)$= 1, $\sup_{t\in\mathbb{R}}\mathcal{L}(t)$ = 0. We shall denote by E the family of all non-distance distribution functions and by G a special non distance distribution function in E given by $G(t) = \begin{cases} 1 & ,\text{if } t < 0 \\ 0 & ,\text{if } t > 0 \end{cases}$

Main Results:

In this section, we introduce the concept of intuitionistic Menger b-metric space with example.

Definition 3.1. The 6-tuple $(X, \mathcal{M}, \mathcal{L}, *, \diamondsuit, b)$ is said to be an intuitionistic Menger b-metric space, if X is an arbitrary set, $b \geq 1$ is given real number, $*$ is a continuous t −norm, $\diamondsuit$ is continuous t −co-norm, $\mathcal{M}$ is a probabilistic distance and $\mathcal{L}$ is a probabilistic non-distance on X satisfying the following conditions: for $\forall\, \alpha, \beta, \gamma \in X$ and $t, s \geq 0$

1. $\mathcal{M}_{(\alpha,\beta)}(t) + \mathcal{L}_{(\alpha,\beta)}(t) \leq 1$;
2. $\mathcal{M}_{(\alpha,\beta)}(0) = 0$;
3. $\mathcal{M}_{(\alpha,\beta)}(t) = \mathcal{H}(\mathrm{t})$, if and only if $\alpha = \beta$;
4. $\mathcal{M}_{(\alpha,\beta)}(t) = \mathcal{M}_{(\beta,\alpha)}(t)$;
5. $\mathcal{M}_{(\alpha,\gamma)}(t+u) \geq \mathcal{M}_{(\alpha,\beta)}\left(\frac{t}{b}\right) * \mathcal{M}_{(\beta,\gamma)}\left(\frac{u}{b}\right)$;
6. $\mathcal{L}_{(\alpha,\beta)}(0) = 1$;
7. $\mathcal{L}_{(\alpha,\beta)}(t) = G(t)$ if and only if $\alpha = \beta$;
8. $\mathcal{L}_{(\alpha,\beta)}(t) = \mathcal{L}_{(\beta,\alpha)}(t)$;
9. $\mathcal{L}_{(\alpha,\gamma)}(t+u) \leq \mathcal{L}_{(\alpha,\beta)}\left(\frac{t}{b}\right) \diamondsuit \mathcal{L}_{(\beta,\gamma)}\left(\frac{u}{b}\right)$;

Definition 3.2. Let $(X, \mathcal{M}, \mathcal{L}, *, \diamondsuit, b)$ be an intuitionistic Menger b-metric space.

(a) A sequence $\{\alpha_n\}$ in X is said to be convergent if there exists $\alpha \in X$ such that $\lim_{n\to\infty} \mathcal{M}_{(\alpha_n,\alpha)}(t) = 1$ and $\lim_{n\to\infty} \mathcal{L}_{(\alpha_n,\alpha)}(t) = 0$, $\forall\, t > 0$. In this case α is called the limit of the sequence $\{\alpha_n\}$ and we write $\lim_{n\to\infty} \alpha_n = \alpha$, or $\alpha_n \to \alpha$.

(b) A sequence $\{\alpha_n\}$ in $(X, \mathcal{M}, \mathcal{L}, *, \diamondsuit, b)$ is said to be a Cauchy sequence if for every $\alpha_n \in X$,there exists $n_0 \in \mathbb{N}$ such that $\mathcal{M}_{(\alpha_n,\alpha_m)}(t) > 1 - \epsilon$ and $\mathcal{L}_{(\alpha_n,\alpha_m)}(t) < \epsilon, \forall\, m, n \geq n_0$ *and* $t > 0$.

(c) The space X is said to be complete if every Cauchy sequence converges in it.

Example 3.3. Let $X = (0, \infty)$. Define a CTN $*: [0,1] \times [0,1] \to [0,1]$ by $g * h = gh$ and a CTCN $\diamondsuit: [0,1] \times [0,1] \to [0,1]$ by $g \diamondsuit h = \max\{g, h\}$ and also define $\mathcal{M}_b$ and $\mathcal{L}_b$ by

$$\mathcal{M}_{b(\alpha,\beta)}(t) = \begin{cases} [e^{\frac{(\alpha-\beta)^2}{t}}]^{-1}, & t > 0 \\ 0 \quad , & t = 0 \end{cases}$$

$$\mathcal{L}_{b(\alpha,\beta)}(t) = \begin{cases} 1 - [e^{\frac{(\alpha-\beta)^2}{t}}]^{-1}, & t > 0 \\ 1 \quad , & t = 0 \end{cases}$$

for all $\alpha, \beta \in X, t > 0$. Then it is an intuitionistic Menger b-metric space.

Proof. Now we prove (5) and (9)

$$(\alpha-\gamma)^2 \leq \left(\frac{b(t+s)}{t}\right)(\alpha-\beta)^2 + \left(\frac{b(t+s)}{s}\right)(\beta-\gamma)^2,$$

where b is an arbitrary integer, we have

$$\frac{(\alpha-\gamma)^2}{b(t+s)} \leq \frac{(\alpha-\beta)^2}{t} + \frac{(\beta-\gamma)^2}{s}.$$

That is, $$e^{\frac{(\alpha-\gamma)^2}{b(t+s)}} \leq e^{\frac{(\alpha-\beta)^2}{t}} . e^{\frac{(\beta-\gamma)^2}{s}}.$$

Since e^e is an increasing function, for $e > 0$, we have

$$[e^{\frac{(\alpha-\gamma)^2}{b(t+s)}}]^{-1} \geq \left[e^{\frac{(\alpha-\beta)^2}{t}}\right]^{-1} . \left[e^{\frac{(\beta-\gamma)^2}{s}}\right]^{-1},$$

$$\mathcal{M}_{b(\alpha,\gamma)} b(t+s) \geq \mathcal{M}_{b(\alpha,\beta)}(t) * \mathcal{M}_{b(\beta,\gamma)}(s)$$

for all $\alpha, \beta, \gamma \in X, t, s > 0$. Hence (5) hold

Since $$(\alpha-\gamma)^2 \geq \max\{(\alpha-\beta)^2, (\beta-\gamma)^2\},$$

$$\frac{(\alpha-\gamma)^2}{b(t+s)} \le \max\left\{\frac{(\alpha-\beta)^2}{t}, \frac{(\beta-\gamma)^2}{s}\right\},$$

where b is an arbitrary integer. Since e^e is an increasing function, for $e > 0$, we have

$$e^{\frac{(\alpha-\gamma)^2}{b(t+s)}} \le \max\left\{e^{\frac{(\alpha-\beta)^2}{t}}, e^{\frac{(\beta-\gamma)^2}{s}}\right\}.$$

So $$[e^{\frac{(\alpha-\gamma)^2}{b(t+s)}}]^{-1} \ge \max\{\left[e^{\frac{(\alpha-\beta)^2}{t}}\right]^{-1}, [e^{\frac{(\beta-\gamma)^2}{s}}]^{-1}\}$$

That is $$1-[e^{\frac{(\alpha-\gamma)^2}{b(t+s)}}]^{-1} \le \max\{1-\left[e^{\frac{(\alpha-\beta)^2}{t}}\right]^{-1}, 1-[e^{\frac{(\beta-\gamma)^2}{s}}]^{-1}\}$$

$$\mathcal{L}_{b(\alpha,\gamma)}b(t+s) \le \mathcal{L}_{b(\alpha,\beta)}(t) * \mathcal{L}_{b(\beta,\gamma)}(s)$$

for all $\alpha, \beta, \gamma \in X, t, s > 0$. Hence (9) hold.

The Banach Contraction Theorem:

In this section, we prove a unique common fixed point theorem which is the extension of Banach contraction theorem in the structure of intuitionistic Menger b-metric space.

Theorem 3.4. Let $(X, \mathcal{M}, \mathcal{L}, *, \diamondsuit, b)$ be a complete intuitionistic Menger b-metric space such that $\lim_{n\to\infty} \mathcal{M}_{b(\alpha,\beta)}(t) = 1$, $\lim_{n\to\infty} \mathcal{L}_{b(\alpha,\beta)}(t) = 0$ for all $\alpha, \beta \in X, t > 0$ and $T: X \to X$ be a mapping satisfying the conditions

$\mathcal{M}_{b(T\alpha,T\beta)}(at) \ge \mathcal{M}_{b(\alpha,\beta)}(t)$ and $\mathcal{L}_{b(T\alpha,T\beta)}(at) \le \mathcal{L}_{b(\alpha,\beta)}(t)$ ……. (3.4.1)

for all $\alpha, \beta \in X, t > 0$, where $a \in (0,1)$. Then T has a unique fixed point $w \in X$ and $\mathcal{M}_{b(w,w)}(t) = 1, \mathcal{L}_{b(w,w)}(t) = 0$, for all $t > 0$.

Proof. Let $(X, \mathcal{M}, \mathcal{L}, *, \diamondsuit, b)$ be a complete intuitionistic Menger b-metric space. For a given element $\alpha_0 \in X$, define a sequence $\{\alpha_n\}$ in X by

$\alpha_1 = T\alpha_0, \alpha_2 = T^2\alpha_0 = T\alpha_1 \ldots\ldots, \alpha_n = T^n\alpha_0 = T\alpha_{n-1}$ for all $n \in \mathbb{N}$
If $\alpha_n = \alpha_{n-1}$ for some $n \in \mathbb{N}$,then α_n is a fixed point of T. We assume that $\alpha_n \ne \alpha_{n-1}$ for all $n \in \mathbb{N}$. For all $t > 0$ and $n \in \mathbb{N}$, we get from (3.4.1) that

$$\mathcal{M}_{b(\alpha_n,\alpha_{n+1})}(t) \ge \mathcal{M}_{b(\alpha_{n+1},\alpha_n)}(at) = \mathcal{M}_{b(T\alpha_n,T\alpha_{n-1})}(at) \ge \mathcal{M}_{b(\alpha_n,\alpha_{n-1})}(at)$$

and

$$\mathcal{L}_{b(\alpha_n,\alpha_{n+1})}(t) \le \mathcal{L}_{b(\alpha_{n+1},\alpha_n)}(at) = \mathcal{L}_{b(T\alpha_n,T\alpha_{n-1})}(at) \le \mathcal{L}_{b(\alpha_n,\alpha_{n-1})}(at)$$

for all $n \in \mathbb{N}$ and $t > 0$. Therefore, by applying the above expression, we can deduce that

$$\mathcal{M}_{b(\alpha_{n+1},\alpha_n)}(t) \ge \mathcal{M}_{b(\alpha_{n+1},\alpha_n)}(at) = \mathcal{M}_{b(T\alpha_n,T\alpha_{n-1})}(at) \ge \mathcal{M}_{b(\alpha_n,\alpha_{n-1})}(t) \ldots\ldots (3.4.2)$$

$$= \mathcal{M}_{b(T\alpha_{n-1},T\alpha_{n-2})}(t) \ge \mathcal{M}_{b(\alpha_{n-1},\alpha_{n-2})}\left(\frac{t}{a}\right) \ge \cdots \ge \mathcal{M}_{b(\alpha_1,\alpha_0)}\left(\frac{t}{a^n}\right)$$

and

$$\mathcal{L}_{b(\alpha_{n+1},\alpha_n)}(t) \le \mathcal{L}_{b(\alpha_{n+1},\alpha_n)}(at) = \mathcal{L}_{b(T\alpha_n,T\alpha_{n-1})}(at) \le \mathcal{L}_{b(\alpha_n,\alpha_{n-1})}(t) \ldots\ldots (3.4.3)$$

$$= \mathcal{L}_{b(T\alpha_{n-1},T\alpha_{n-2})}(t) \le \mathcal{L}_{b(\alpha_{n-1},\alpha_{n-2})}\left(\frac{t}{a}\right) \le \cdots \le \mathcal{L}_{b(\alpha_1,\alpha_0)}\left(\frac{t}{a^n}\right)$$

for all $n \in \mathbb{N}, p \geq 1$ and $t > 0$. Thus, we have

$$\mathcal{M}_{b(\alpha_n,\alpha_{n+1})}(t) \geq \mathcal{M}_{b(\alpha_n,\alpha_{n+1})}\left(\frac{t}{b}\right) * \mathcal{M}_{b(\alpha_{n+1},\alpha_{n+p})}\left(\frac{t}{b}\right)$$

and

$$\mathcal{L}_{b(\alpha_n,\alpha_{n+1})}(t) \leq \mathcal{L}_{b(\alpha_n,\alpha_{n+1})}\left(\frac{t}{b}\right) \diamondsuit \mathcal{L}_{b(\alpha_{n+1},\alpha_{n+p})}\left(\frac{t}{b}\right).$$

Continuing in this way, we get

$$\mathcal{M}_{b(\alpha_n,\alpha_{n+p})}(t) \geq \mathcal{M}_{b(\alpha_n,\alpha_{n+1})}\left(\frac{t}{b}\right) * \mathcal{M}_{b(\alpha_{n+1},\alpha_{n+2})}\left(\frac{t}{b^2}\right) * \dots * \mathcal{M}_{b(\alpha_{n+p-1},\alpha_{n+p})}\left(\frac{t}{b^{p-1}}\right)$$

and

$$\mathcal{L}_{b(\alpha_n,\alpha_{n+p})}(t) \leq \mathcal{L}_{b(\alpha_n,\alpha_{n+1})}\left(\frac{t}{b}\right) \diamondsuit \mathcal{L}_{b(\alpha_{n+1},\alpha_{n+2})}\left(\frac{t}{b^2}\right) \diamondsuit \dots \diamondsuit \mathcal{L}_{b(\alpha_{n+p-1},\alpha_{n+p})}\left(\frac{t}{b^{p-1}}\right)$$

Using (3.4.2) and (3.4.3) in the above inequality, we have

$$\mathcal{M}_{b(\alpha_n,\alpha_{n+p})}(t) \geq \mathcal{M}_{b(\alpha_0,\alpha_1)}\left(\frac{t}{ba^n}\right) * \mathcal{M}_{b(\alpha_0,\alpha_1)}\left(\frac{t}{b^2a^{n+1}}\right) * \dots * \mathcal{M}_{b(\alpha_0,\alpha_1)}\left(\frac{t}{b^{p-1}a^{n+p-1}}\right) \dots (3.4.4)$$

and

$$\mathcal{L}_{b(\alpha_n,\alpha_{n+p})}(t) \leq \mathcal{L}_{b(\alpha_0,\alpha_1)}\left(\frac{t}{ba^n}\right) \diamondsuit \mathcal{L}_{b(\alpha_0,\alpha_1)}\left(\frac{t}{b^2a^{n+1}}\right) \diamondsuit \dots \diamondsuit \mathcal{L}_{b(\alpha_0,\alpha_1)}\left(\frac{t}{b^{p-1}a^{n+p-1}}\right) \dots (3.4.5)$$

Here b is an arbitrary positive integer.

We know that $\lim_{n\to\infty} \mathcal{M}_{b(\alpha,\beta)}(t) = 1$, $\lim_{n\to\infty} \mathcal{L}_{b(\alpha,\beta)}(t) = 0$ for all $\alpha, \beta \in X, t > 0$, where $a \in (0,1)$. It follows from (3.4.4) and (3.4.5) that

$$\lim_{n\to\infty} \mathcal{M}_{b(\alpha_n,\alpha_{n+p})}(t) = 1 * 1 * \dots * 1 = 1 \text{ for all } t > 0, p \geq 1$$

and

$$\lim_{n\to\infty} \mathcal{L}_{b(\alpha_n,\alpha_{n+p})}(t) = 0 \diamondsuit 0 \diamondsuit \dots 0 = 0 \text{ for all } t > 0, p \geq 1.$$

Hence $\{\alpha_n\}$ is a Cauchy sequence.

The completeness of the intuitionistic Menger b-metric space $(X, \mathcal{M}, \mathcal{L}, *, \diamondsuit, b)$ ensures that there exists $w \in X$ such that

$$\lim_{n\to\infty} \mathcal{M}_{b(\alpha_n,w)}(t) = \lim_{n\to\infty} \mathcal{M}_{b(\alpha_n,\alpha_{n+p})}(t) = \mathcal{M}_{b(w,w)}(t) = 1, \text{ for all } t > 0, p \geq 1 \dots (3.4.6)$$

and

$$\lim_{n\to\infty} \mathcal{L}_{b(\alpha_n,w)}(t) = \lim_{n\to\infty} \mathcal{L}_{b(\alpha_n,\alpha_{n+p})}(t) = \mathcal{L}_{b(w,w)}(t) = 0. \text{ for all } t > 0, p \geq 1 \dots (3.4.7)$$

Now, we show that $w \in X$ is a fixed point of T. We have

$$\begin{aligned} \mathcal{M}_{b(w,Tw)}(t) &\geq \mathcal{M}_{b(w,\alpha_{n+p})}\left(\frac{t}{2b}\right) * \mathcal{M}_{b(\alpha_{n+1},Tw)}\left(\frac{t}{2b}\right) \\ &= \mathcal{M}_{b(w,\alpha_{n+1})}\left(\frac{t}{2b}\right) * \mathcal{M}_{b(T\alpha_n,Tw)}\left(\frac{t}{2b}\right) \\ &\geq \mathcal{M}_{b(w,\alpha_{n+1})}\left(\frac{t}{2b}\right) * \mathcal{M}_{b(\alpha_n,w)}\left(\frac{t}{2ba}\right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_{b(w,Tw)}(t) &\leq \mathcal{L}_{b(w,\alpha_{n+p})}\left(\frac{t}{2b}\right) \diamondsuit \mathcal{L}_{b(\alpha_{n+1},Tw)}\left(\frac{t}{2b}\right) \\ &= \mathcal{L}_{b(w,\alpha_{n+1})}\left(\frac{t}{2b}\right) \diamondsuit \mathcal{L}_{b(T\alpha_n,Tw)}\left(\frac{t}{2b}\right) \\ &\leq \mathcal{L}_{b(w,\alpha_{n+1})}\left(\frac{t}{2b}\right) \diamondsuit \mathcal{L}_{b(\alpha_n,w)}\left(\frac{t}{2ba}\right) \end{aligned}$$

For all $t > 0$. Taking the limit as $n \to \infty$, and by (3.4.6) and (3.4.7), we get

$$\mathcal{M}_{b(w,Tw)}(t) = 1 * 1 = 1$$

and

$$\mathcal{L}_{b(w,Tw)}(t) = 0\Diamond 0 = 0.$$

Therefore, w is a fixed point of T and $\mathcal{M}_{b(w,w)}(t) = 1, \mathcal{L}_{b(w,w)}(t) = 0$, for all , $t > 0$. Now, we investigate the uniqueness of fixed point. For this, assume that v and w are two fixed point of T. Then by (3.4.1), we have

$$\mathcal{M}_{b(w,v)}(t) = \mathcal{M}_{b(Tw,Tv)}(t) \geq \mathcal{M}_{b(w,v)}\left(\frac{t}{a}\right)$$

and

$$\mathcal{L}_{b(w,v)}(t) = \mathcal{L}_{b(Tw,Tv)}(t) \leq \mathcal{L}_{b(w,v)}\left(\frac{t}{a}\right)$$

for all $t > 0$. Thus, we obtain

$\mathcal{M}_{b(w,v)}(t) \geq \mathcal{M}_{b(w,v)}\left(\frac{t}{a^n}\right)$ for all $n \in \mathbb{N}$

and

$\mathcal{L}_{b(w,v)}(t) \leq \mathcal{L}_{b(w,v)}\left(\frac{t}{a^n}\right)$ for all $n \in \mathbb{N}$

Taking the limit as $n \to \infty$ and using $\lim_{n\to\infty} \mathcal{M}_{b(\alpha,\beta)}(t) = 1$, $\lim_{n\to\infty} \mathcal{L}_{b(\alpha,\beta)}(t) = 0$,we get $w = v$. Hence the fixed point is unique.

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