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Some fixed point theorem using non- expansive mapping in b-metric space

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Abstract

In this paper, we establish fixed point results for a class of nonexpansive self-mappings defined on complete b-metric spaces. Motivated by the work of Mohammadi, Golkarmanesh, and Parvaneh[7], the present study extends and generalizes existing fixed point results for nonexpansive mappings under suitable contractive conditions. The main results are obtained for mappings defined on a b-metric space endowed with a binary relation satisfying an appropriate invariance condition. By constructing a Picard sequence from a suitable initial point, we prove that the resulting sequence is Cauchy and converges to a fixed point in the complete b-metric space. Furthermore, the results provide conditions concerning the existence and separation of fixed points. The proposed results extend earlier findings and contribute to the development of fixed point theory by incorporating the generalized structure of b-metric spaces and weaker contractive conditions. These results may be useful in the study of nonlinear analysis and mathematical models in which the standard metric space framework is insufficient[6,5].

Keywords

Fixed point theory; Nonexpansive mapping; ***b****-metric space; Banach contraction principle; Picard sequence; Contractive mapping; Fixed point theorem; Complete* ***b****-metric space; Self-mapping.*

Introduction

This paper, motivated by the work of B.mohammadi[7], we establish new fixed point theorems for nonexpansive mappings in the setting of b-metric spaces. Our results not only extend and generalize the findings of Khojasteh et al. (2014)[2] but also provide a broader perspective on the existence and uniqueness of fixed points under weaker contractive conditions. The significance of these results lies in their applicability to various mathematical models and real-world problems where standard metric space assumptions do not hold.

Assume that (X, d) be a metric space and $f: X \to X$ be a single-valued mapping. Then, f is called a k-Lipscitz mapping if $d(fx, fy) \leq d(x, y)$.

for all $x, y \in X$,where $k \geq 0$. If $k \in [0,1)$, then it is called a contractive mapping and if k =1, f is called a nonexpansive mapping. In Banach[1] proved a very important fixed point result for a contractive self-mapping. Obviously, any contractive mapping is nonexpansive, but there verse is not true in general .Several researchers investigated fixed point theory for nonexpansive mappings such as [6,5]. In [2] Khojasteh et al. investigated fixed point results for a new type of self mappings and multivalued mappings. Then,Vetro in [3] extended their results for nonexpansive mappings using a binary relation on X, called f-invariant. Also, he established fixed point results for nonexpansive multi-valued mappings with a new type of contraction. On the other hand, fixed point theory for mappings on b-metric spaces as a generalization of metric spaces has attracted many researchers for many years ([9,10,11,12,13,14)]).

Preliminaries-

Preliminaries give some notations and results that will be needed in the sequel.

Definition[4]

$Let\ X\ be\ a\ nonexpansive\ set\ and\ s \geq 1\ be\ a\ constant\ real\ number. The\ function$

$d: X \times X \to [0, \infty) is\ called\ a\ metric\ on\ X\ if\ the\ \ following\ conditions\ hold:$

$(i) d(x, y) = 0\ iff\ x = y\ for\ all\ x, y\ \in X.$

$(ii) d(x,y) = d(y,x)$ *for all* $x, y \in X$
$(iii) d(x,y) \leq s[d(x,z) + d(z,y)]$ *for all* $x, y, z \in X$.
In this case, (X, d) is called a b-metric space with parameter s.
Denote by CB(X) the set of all nonempty closed bounded subsets of X.
Assume that H be the Pompeiu – Hausdorff metric on CB(X) defined by

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\},$$

For all A,B ∈ CB(X), where d(x, B) = $\inf_{y \in B} d(x,y)$. An element $x \in X$ is said to be a fixed point of a multi –valued mapping T: X →CB(X) whenever x ∈ Tx. It is said that T : X→CB(X) is contractive whenever H(Tx, Ty)≤ kd(x,y) for all x , y $\in X$, *where* $k \in [0, 1)$. If k = 1, then T is called a nonexpansive multivalued mapping. Nadler[8] proved the existence of fixed point for contractive multivalued mappings.

Lemma[3]

If $\{a_n\}$ *be a nonincreasing sequence of nonnegative real numbers, then the sequence* $\{\frac{a_n + a_{n+1}}{a_n + a_{n+1} + 1}\}$.

Corollary[7]

Let (X, d) *be a* $b-$*metric space and* $f: X \to X$ *be a nonexpansive mapping*.
If $x_0 \in X$ *and* $\{x_0\}$ *be a Picard sequence starting with* x_0, *that is*, $x_n = f x_{n-1}$ *for all* $n \in N$ *then, the sequence*

$$\{\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1}\}$$

is increasing too.

Main Result:

We prove some results for single-valued mappings defined on a b-metric space endowed with relation on X , that is, R is subset of $X \times X$. Then, R is Banach f-invariant if $(fx, f^2 x) \in R$, whenever $(x, fx) \in R$. Also, a subset Y of X is well ordered with respect to R if for all $x, y \in Y$ we have $(x,y) \in R$ or $(y,x) \in R$. Let $Fix(f) = \{x \in X : x = fx\}$ denotes the set of all fixed points of f on X.

Theorem:

Let (X, d) *be a complete* $b-$*metric space with parameter* $s > 1$ *endowed with a binary relation* R *on* X *and* $f: X \to X$ *be a nonexpansive mapping such that*

$$d(fx, fy) \leq \left(\frac{d(x, fy) + d(y, fx)}{s[d(x, fx) + d(y, fy) + 1]} + k\left[\frac{d(x, fy) + d(y, fx)}{s}\right]\right) d(x,y) \qquad (1)$$

A *for all* $(x,y) \in R$, *where* $k \in [0,1)$. *Also assume that*
$(a) R$ *is Banach* $f-$*invarient*,
(b)*if* $\{a_n\}$ *is a sequence in* X *such that* $(x_{n-1}, x_n) \in R$ *for all* $n \in N$ *and* $x_n \in X$ *as* $n \to \infty$, *as* $n \to \infty$, *then* $(x_{n-1}, z) \in R$, *for all* $n \in N$.
$(c) Fix(f)$ *is well ordered with respect to* R.
Let there exists $x_0 \in X$ *such that* $(x_0, fx_0) \in R$ *and* $\varphi s < 1$, *where*

$$\varphi = \left(\frac{d(x_0, fx_0) + d(x_0, f^2 x_0)}{d(x_0, fx_0) + d(fx_0, f^2 x_0) + 1} + k[d(x_0, fx_0) + d(x_0, f^2 x_0)]\right)$$

Then,
(i)f *has at least one fixed point* $z \in X$,
(ii)*the Picard sequence of initial point* $x_0 \in X$ *converges to a fixed point of* f.

Proof:

Let $x_0 \in X$ be such that $(x_0, fx_0) \in R, \varphi s < 1$ and let $\{x_n\}$ be a Picard sequence with initial point x_0. If $x_{n-1} = x_n$ for some $n \in N$, then x_{n-1} is a fixed point of f and the existence of a fixed point is proved. Now, we suppose that $x_{n-1} \neq x_n$ for all $n \in N$. From $(x_0, x_1) = (x_0, fx_0) \in R$, since R is Banach f-invarient, we deduce that $(x_1, x_2) = (fx_0, f^2 x_0) \in R$. This implies that $(x_{n-1}, x_n) = (f^{n-1} x_0, f^n x_0) \in R$ for all $n \in N$. Using the contractive condition (1) with $x = x_{n-1}$ and $y = x_n$, we get

$$d(x_n,x_{n-1}) = d(fx_{n-1},fx_n) \le \Big(\frac{d(x_{n-1},fx_n)+d(x_n,fx_{n-1})}{s[d(x_{n-1},fx_{n-1})+d+(x_n,fx_n)+1]} + k\left[\frac{d(x_{n-1},fx_n)+d(x_n,fx_{n-1})}{s}\right]\Big)d(x_{n-1},x_n)$$
$$= \left(\frac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{s[d(x_{n-1},x_n)+d(x_n,fx_{n-1})]} + k\left[\frac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{s}\right]\right) d(x_{n-1},x_n)$$
$$\le \left(\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{[d(x_{n-1},x_n)+d(x_n,x_{n+1})+1]} + k[d(x_0,fx_0)+d(x_0,f^2x_0)]\right) d(x_{n-1},x_n) \tag{2}$$

For all $n \in N$. From (2), by Corollary, we get

$$d(x_n,x_{n+1}) \le \Big(\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{[d(x_{n-1},x_n)+d(x_n,x_{n+1})+1]} + k[d(x_{n-1},x_n)+d(x_n,x_{n+1})]\Big)d(x_{n-1},x_n)$$
$$\le\Big(\frac{d(x_0,x_1)+d(x_1,x_2)}{[d(x_0,x_1)+d(x_1,x_2)+1]} + k[d(x_0,x_1)+d(x_1,x_2)]\Big)d(x_{n-1},x_n) \tag{3}$$
$$= \varphi d(x_{n-1},x_n),$$

For all $n \in N$. Thus, for any m,n $\in N$ with $m > n$, we have

$$d(x_n,x_m) \le s[d(x_n,x_{n+1}) + d(x_{n+1},x_m)]$$
$$\le sd(x_n,x_{n+1}) + s[d(x_{n+1},x_{n+2}) + d(x_{n+2},x_m)]$$
$$\le sd(x_n,x_{n+1}) + s^2d(x_{n+1},x_{n+2}) + \cdots + s^{m-n-1}d(x_{m-1},x_m)$$
$$=\sum_{i=1}^{m-n-1} s^i\, d(x_{n+i-1},x_{n+i}) \le \sum_{i=1}^{m-n-1} s^i\, \varphi^{i+n-1} d(x_0,x_1)$$
$$=\varphi^{n-1}d(x_0,x_1)(\varphi s)^{1-n}\sum_{i=n}^{m-1}(\varphi s)^i$$
$$=d(x_0,x_1)s^{1-n}\sum_{i=n}^{m-1}(\varphi s)^i \tag{4}$$

As $\varphi s < 1$ and $s > 1$, the last term in the above tends to zero, as $m,n \to \infty$. Thus, $\{x_n\}$ is a Cauchy sequence, Since (X,d) is complete, there exists $z \in X$ such that $x_n \to z$. Now we show that z is a fixed point of f. By assumption (b), we deduce that $(x_n,z) \in R$. So, by (1),
We have

$$d(x_{n+1},fz) = d(fx_n,fz) \le \Big(\frac{d(x_n,fz)+d(z,fx_n)}{s[d(x_n,fx_n)+d(z,fz)+1]} + k\left[\frac{d(x_n,fz)+d(z,fx_n)}{s}\right]\Big)d(x_n,z)$$
$$=\left(\frac{d(x_n,fz)+d(z,x_{n+1})}{s[d(x_n,x_{n+1})+d(z,fz)+1]} + k\left[\frac{d(x_n,fz)+d(z,x_{n+1})}{s}\right]\right) d(x_n,z)$$

Taking limit as $n \to \infty$, in the above inequality we get, $d(z,fz) \le 0$. Thus $(z,fz) = 0$, that is $z = fz$. Thus (i) and (ii) hold.

Conclusion:

The study establishes the existence of fixed points for a certain class of nonexpansive mappings in a Banach space. The authors demonstrate that under specific contractive conditions, a self-mapping function possesses at least one fixed point. Furthermore, they show that the Picard sequence generated by an initial point in the space converges to a fixed point. Additionally, the results ensure that if two distinct fixed points exist, their distance is bounded below by a derived expression. Above given main result is proven by multiplying inequality with constant k ,which is the extension of B.Mohammadi[7]. These findings contribute to the broader field of fixed point theory, extending previous results by incorporating new constraints on nonexpansive mapping.

References:

[1] Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundamenta Mathematicae, 3(1), 133–181.3

[2] F.Khojasteh ,M.Abbas,and S.Costache, Two new types of fixed point theorems in complete metric spaces, Abstract and Applied Analysis, 2014 (2014), ArticleID 325840, 10

[3] F.Vetro, Fixed point results for nonexpansive mappings on metric spaces, Filomat,29(2015),2011–2020.11

[4] I.A. Bakhtin, The contraction mapping principle in almost metric Space,(Russian) Func. Anal., Ulyanovsk. Gos. Ped. Inst. ,Ulyanovsk, 30 (1989),26-37.17

[5] M. A. Khamsi and S.Reich ,Nonexpansive mappings and semigroups in hyper convex spaces, Math. Japonica, 35(1990), 467–471. 22

[6] M.E delstein, On nonexpansive mappings, Proc. Amer. Math. Soc., 15 (1964), 689–695.24

[7] Mohammadi B.,Golkarmanesh F.& Parvaneh V." Some Fixed Point Theorems for

Nonexpansive Self-Mappings and Multi-Valued Mappings in b-Metric Spaces” Journal of Mathematical Extension Vol. 14,No. 1,(2020),1-18.25

[8] N. Haokip and N. Goswami, Some fixed point theorems for generalized Kannan type mappings in b-metric spaces, Proyecciones Journal of Mathematics 38 (4), 763–782, (2019). 26

[9] S.Aleksi´ c,H.Huang, Z.D. Mitrovi´ c, and S. Radenovi´ c, Remarks on some fixed point results in b-metric spaces, J.Fixed PointTheory Appl.,(2018), 20:147,https://doi.org/10.1007/s11784-018-0626-2. 32

[10] S.Aleksi´c,T.Do.enovi´c,Z.D.Mitrovi´ c, and S.Radenovi´ c, Remarks on common fixed point results for generalized α-ψ-contraction multivalued mappings in b-metric spaces, to appear in Adv. Fixed PointTheory, 9(1) (2009), 1-16. 33

[11] S.Chandok, M. Jovanovic, and S.Radenovic, Ordered b-metric spaces and Geraghty type contractive mappings,Vojnotehnickiglasnik/ Military TechnicalCourier,65(2)(2017),331-345.35

[12] T.Dosenovi´c, M.Pavlovi´c, and S.Radenovi´ c, Contractive conditions in b-metricspaces,Vojnotehnickiglasnik /Military Technical Courier, 65 (4) (2017), 851-865.39

[13] Z. Mustafa ,J.R .Roshan, V.Parvaneh, and Z.Kadelburg, Some common fixed point results in ordered partial b-metric spaces, J. Inequal .Appl., (2013), 26

[14] Z.Kadelburg, Lj. Paunovi´c, and S.Radenovi´ c, Anote on fixed point theorems for weaklyT-Kannanand weakly T-Chatterje a contractions in b-metricspaces, Gulf J.Math., 3(3) (2015), 57-67.46